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ELEMENTS

OF

ALGEBRA

BY

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WITH AN APPENDIX

CONTAINING MISCELLANEOUS INVESTIGATIONS, AND A COLLECTION OF PROBLEMS IN VARIOUS PARTS OF ALGEBRA

Sixth Edition, Corrected

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LUDGATE HILL

1871

PREFACE TO THE FOURTH EDITION.

Every one who has made some advancement in the study of this important branch of mathematical science, must have found that, until he had acquired a knowledge of the nature of Equations, his progress was uninteresting, and, consequently, unsatisfactory to himself, because he met with many things which were not easily comprehensible to him, and because the application of the principles he was endeavouring to comprehend was so remote that he would frequently be led to question the utility of this science. This circumstance has frequently operated as a serious discouragement to persons entering upon the study of Algebra.

The arrangement of the following pages is such that, as soon as the student shall have made himself so far acquainted with the first principles as to be able to perform the ordinary operations of Addition, Subtraction, Multiplication, and Division, he shall be invited to consider the nature of an Equation, and to apply his knowledge, however small, to the *solution of a Problem*. Again, when he has become conversant with Algebraic Fractions, he will be presented with other equations and problems, which it is hoped his increased knowledge will enable him to encounter with confidence and success.

One of the greatest of modern mathematicians has said that "In ediscendis scientiis exempla plus prosunt quam præcepta;" and any one who has been long practically engaged in the work of education will readily agree with him. With a view to assist the student, explanations and solutions of many examples and problems are given in all parts of this work.

This Fourth Edition has been carefully revised, and such corrections and alterations introduced as will, it is hoped,

render the work more acceptable to both teachers and learners. The encouragement it has already received has induced the publisher to recommend the enlargement of the work to the extent of an additional sheet: this has been filled up with the new matter forming the APPENDIX at the end. Opportunity has thus been afforded to introduce several topics of interest and utility that could not well be treated of in the body of the work, and at the same time to furnish the student with a collection of problems suited to exercise his talents and ingenuity in a tolerably extensive range of subjects in elementary algebra.

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ALGEBRA.

CHAPTER I

1 ALGEBRA is a science in which numerical magnitude in general is represented by symbols and signs.

The *symbols* are a, b, c, &c., or α β γ, &c., to represent *known* quantities, and x, y, z, &c., or θ, ϕ, ψ, &c., to represent *unknown* quantities.

The signs are

$=$ *equal.*

$+$ *plus*, for *Addition;* thus, $a + b =$ the *sum* of a and b, just as $3 + 5 =$ the sum of 3 and $5 = 8$.

$-$ *minus*, for *Subtraction;* thus $b - c =$ the excess of b above c, just as $5 - 2 =$ the excess of 5 above $2 = 3$, When it is not known which of two quantities, as x and y, is the greater, their *difference* is expressed thus, $x \sim y$.

$\times$ or $\cdot$ for *Multiplication*, thus $2\,c \times 3\,d =$ the *product* of twice c and thrice d. The 2 is called the *coefficient* of c, and the 3 the coefficient of d. In the expressions $a\,x$ and $b\,x$, that is, $a \times x$ and $b \times x$, a and b are the *literal* coefficients of x. The product is sometimes expressed thus, $(a + b)\,.\,(c - d)$ or $(a + b)\,(c - d)$ or $\overline{a + b} \cdot \overline{c - d}$, which means the sum of a and b *multiplied* by the difference of c and d. The expression $a\,b + c$ means that a is *multiplied* by b, and then c is added to the

product; if $a = 2$, $b = 3$, and $c = 1$, then $a b + c = 2 \times 3 + 1 = 7$. The expression $a(b + c)$ means that b and c are added together, and then their sum is multiplied by a; if $a = 2$, $b = 3$, and $c = 1$, as before, then $a(b + c) = 2 \times (3 + 1) = 2 \times 4 = 8$.

$\div$ for *Division;* thus $m \div n$, or more commonly $\frac{m}{n}$, expresses the division of m by n; if $m = 6$ and $n = 2$, then $\frac{m}{n} = \frac{6}{2} = 3$.

: *is to*, and :: *as* or *so is*, for *Ratios;* thus $a : b :: c : d$ means that a has to b the *same ratio* that c has to d.

$\surd$ or $\sqrt{}$ the *radical* sign, for roots; thus $\sqrt{x}$ means the *square root* of x, $\sqrt[3]{x+y}$, or $\sqrt[3]{(x + y)}$ the *cube root* of the sum of x and y.

${}^{2}, {}^{3}, {}^{4}$, &c. *indices*, or *exponents*, for *powers;* thus x^2 means the *square* or *second power* of x, $(y - z)^4$ the *fourth power* of the difference of y and z; if $y = 6$ and $z = 4$, then $(y - z)^4 = (6 - 4)^4 = 2^4 = 16$.

$\because$ *since* or *because;* $\therefore$ *therefore;* $\propto$ *varies as;* $>$ *greater than;* $<$ *less than;* ∞ *infinity;* and $\pm$ *plus* or *minus.*

Example 1. Find the value of the following expressions, when $a = 3$, $b = 2$, and $c = 4$.

1. $a + 3b + c$. Ans. 13.
2. $4a + 2b - ac$. Ans. 4.
3. $a^2 + b(c - a)$. Ans. 11.
4. $5(a + b^3 - 2c)$. Ans. 15.

Ex. 2. Find the value of the following expressions, when $x = 4$, $y = 3$, $z = 1$, and $n = 2$

1. $n(x + y - z)$. Ans. 12.

2. $\dfrac{x^2 + y^3 - 9z}{8n + 1}$. Ans. 2.

3. $\dfrac{x^3}{8} - \dfrac{y^2}{3} + nz$ Ans. 7

4. $(x^2 - y^2) - (x - y)^2 + n(xy - z)$. Ans. 28.

II. *Like* quantities consist of the *same letters*, as $2a$, $5a$; also $3x^2$, $4x^2$; also axy, $2axy$, $10axy$.

Unlike quantities consist of different letters, as $2a$, $2b$, $5ay$.

Positive quantities are such as have the sign $+$, or no sign, before them; in which latter case $+$ is understood.

Negative quantities are such as have the sign $-$ before them.

Suppose a person's money is 2 pounds:
He has a *positive* stock of 2 pounds, which may be represented by $+ 2$.

Let him spend 1 pound.
He has then a *positive* stock of 1 pound, which may be represented by $+ 1$.

Let him again spend 1 pound.
He has then *no positive* stock, which may be represented by 0.

Let him incur a debt of 1 pound.
He has then a *negative* stock of 1 pound, which may be represented by $- 1$.

Let him again incur a debt of 1 pound.
He has then a *negative* stock of 2 pounds, which may be represented by $- 2$.

III. Every algebraic quantity has a *numeral* coefficient either expressed or understood; if it be not expressed, it is 1. Hence the sum of $3a$, $5a$, and a, is $3a + 5a + 1a = 9a$, just as 3 *horses*, 5 *horses*, and a *horse*, are equal to 9 horses. Hence, also, $10z - z = 10z - 1z = 9z$, just as 10 sacks of corn diminished by a sack of corn are equal to 9 sacks of corn.

ADDITION.

Examples.

1. Add together $2a + b$, $3a + 2b$, $a + 5b$, and $3b + 4a$.

$$\begin{array}{rcr} 2a & + & b \\ 3a & + & 2b \\ a & + & 5b \\ 4a & + & 3b \\ \hline 10a & + & 11b. \end{array} \quad \text{Ans.}$$

Here $4a + 1a + 3a + 2a = 10a$
and $3b + 5b + 2b + 1b = 11b$
$\therefore$ the whole sum is $10a + 11b$.

2. Find the sum of $3x - 2y$, $6x + 4y$, $-x - 3y$, and $-y - 5x$.

$$\begin{array}{rcr} 3x & - & 2y \\ 6x & + & 4y \\ -\ x & - & 3y \\ -5x & - & y \\ \hline 3x & - & 2y. \end{array} \quad \text{Ans.}$$

Here $3x + 6x - 1x - 5x = 9x - 6x = 3x$
and $4y - 2y - 3y - 1y = 4y - 6y = -2y$,
for $-6y = -4y - 2y$
$\therefore 4y - 6y = 4y - 4y - 2y = 0 - 2y = -2y$
$\therefore$ the whole sum is $3x - 2y$.

3. Add together $x^2 - 3xy + 2y^2$, $y^2 - 4x^2 + xy$, and $xy - 5y^2 + 3x^2$.

$$\begin{array}{rcrcr} x^2 & - & 3xy & + & 2y^2 \\ -4x^2 & + & xy & + & y^2 \\ 3x^2 & + & xy & - & 5y^2 \\ \hline & - & xy & - & 2y^2 \end{array}$$

Here $x^2 + 3x^2 - 4x^2 = 4x^2 - 4x^2 = 0$
$1xy + 1xy - 3xy = 2xy - 3xy = -1xy$
$2y^2 + 1y^2 - 5y^2 = 3y^2 - 5y^2 = -2y^2$
$\therefore$ the whole sum is $-xy - 2y^2$.

4 Add together $2a + 2b$, $a + 3b$, $5a + b$, and $8a + b$. Ans. $16a + 7b$.

5. Add together $2x - 4y$, $-3x + y$, $6x - 5y$, and $2y - x$. Ans. $4x - 6y$.

6. Find the sum of $2x^2 + xy - 2y^2$, $3xy - y^2 - 4x^2$, $5y^2 - x^2 - 6xy$, and $4x^2 - xy + 3y^2$. Ans. $x^2 - 3xy + 5y^2$.

7. Collect into one sum $ax^3 + bx^2 - cx$, $2ax^3 + 4cx - 5bx^2$, and $2bx^2 - ax^3 - 8cx$. Ans. $2ax^3 - 2bx^2 - 5cx$.

8. Add $a - b$ to $a + b$; $x^2 + xy$ to $xy + y^2$; and $y^2 - xz$ to $xz - z^2$ Ans. $2a$; $x^2 + 2xy + y^2$; and $y^2 - z^2$.

SUBTRACTION.

IV. Suppose A's money is $3x$ pounds, which remains constantly the same; and suppose B's money is $2x$ pounds, which is diminished daily by the expenditure of x pounds (Art. II.); it is required to find the excess of A's money above B's, on each successive day.

	1st day.	2nd day.	3rd day.	4th day.	5th day.
A's money	$3x$	$3x$	$3x$	$3x$	$3x$
B's money	$2x$	$1x$	$0x$	$-1x$	$-2x$
Difference	$1x$	$2x$	$3x$	$4x$	$5x$, &c.

These differences, which show the daily excess of A's money above B's, are obtained by *Subtraction;* but they may be obtained by algebraic *Addition*, if the *signs* of the quantities to be subtracted be previously *changed:* thus,

	1st day.	2nd day.	3rd day.	4th day.	5th day.
A's money	$3x$	$3x$	$3x$	$3x$	$3x$
B's money	$-2x$	$-1x$	$-0x$	$1x$	$2x$
Sum	$1x$	$2x$	$3x$	$4x$	$5x$, &c.

Hence, to *subtract* algebraic quantities; *Rule.* Imagine the signs of the quantities to be subtracted *changed*, and then proceed as in Addition.

V. The sign $-$ before any quantity indicates that the whole of that quantity is to be *subtracted.* Thus the expression $a - (b + c)$ means that $(b + c)$ is to be subtracted from a: hence, $a - (b + c) = a - b - c$, $a - (b - c + d) = a - b + c - d$, and $x - \{a - (b - c)\} = x - \{a - b + c\} = x - a + b - c$.

Examples.

1 From $4a - 8b - 2c$ subtract $a + 2b - 5c$.

$$\begin{array}{l} 4a - 8b - 2c \\ \ \ a + 2b - 5c \\ \hline 3a - 10b + 3c \text{ Ans.} \\ 4a - \ 8b - 2c \\ \hline \end{array}$$

Here we say $+4a - 1a = 3a$.
$-8b - 2b = -10b$.
$-2c + 5c = +3c$.

Proof, by adding the ans. to the expression above it.

2. From $2mx^2 - nx - 3py + r$ subtract $5mx^2 - 4nx + py + q$.

$$\begin{array}{l} 2mx^2 - nx - 3py + r \\ 5mx^2 - 4nx + \ py + q \\ \hline -3mx^2 + 3nx - 4py + r - q \text{ Ans.} \\ \hline \end{array}$$

Here we say
$+2mx^2 - 5mx^2 = -3mx^2$
$-nx + 4nx = +3nx$
$-3py - 1py = -4py$
and $+r - q = +r - q$.

3. From $6a + 2b - (3a + b)$ subtract $2a + 4b - (4a - b)$.

$$\begin{array}{l} 6a + 2b - 3a - b \\ 2a + 4b - 4a + b \\ \hline 4a - 2b + a - 2b = \\ 5a - 4b \quad \text{Ans.} \end{array}$$

Here, in writing down $-(3a + b)$ and $-(4a - b)$, we take care to change the signs within the brackets, because the sign before each bracket is $-$.

Examples, to be proved by Addition.

4. From $5a - 6b - 3c$ subtract $2a + 3b - 7c$.
Ans. $3a - 9b + 4c$.

5. From $x^2 - xy$ take $xy - y^2$; and from $a + b$ take $a - b$.
Ans. $x^2 - 2xy + y^2$; and $2b$.

6. From $5n^2 + n - 3$ take $4n^2 - 3n - 2$.
Ans. $n^2 + 4n - 1$.

7. From $ay^2 - 7a^2y - a - c$ take $3ay^2 - 2a^2y + a - b$.
Ans. $-2ay^2 - 5a^2y - 2a + b - c$.

8. What is the difference between $a^3 - a^2b + 2ab^2 - b^3$ and $2a^2b - ab^2$?
Ans. $a^3 - 3a^2b + 3ab^2 - b^3$.

MULTIPLICATION.

VI. Multiply $3 + 2$ by 4.

$3 + 2 = 5$, and $5 \times 4 = 20 =$ product required; or thus, $(3 + 2) \times 4 = 3 \times 4 + 2 \times 4 = 12 + 8 = 20 =$ product required.

Multiply $a + b$ by c, and also by d, and add the two results

$$\begin{array}{l} (a + b) \times c = ac + bc \\ (a + b) \times d = ad + bd \\ \hline \qquad ac + bc + ad + bd. \text{ Ans.} \end{array}$$

This is obviously the same thing as multiplying $a + b$ by $c + d$; thus

$$\begin{array}{l} a + b \\ c + d \\ \hline ac + bc \\ \qquad + ad + bd \\ \hline ac + bc + ad + bd \\ \hline \end{array}$$

where $a + b$ is multiplied by c and then by d, and the two results added together.

Multiply $5 - 3$ by 6.

$5 - 3 = 2$, and $2 \times 6 = 12 =$ product required;
or thus, $(5 - 3) \times 6 = 30 - 18 = 12 =$ product required.

Multiply $a - b$ by c, and also by d, and subtract the latter result from the former.

$$\begin{array}{l} (a - b) \times c = ac - bc \\ (a - b) \times d = ad - bd \\ \hline \qquad ac - bc - ad + bd. \end{array}$$

This is the same thing as multiplying $a - b$ by $c - d$; thus

$$\begin{array}{l} a - b \\ c - d \\ \hline ac - bc \\ \qquad - ad + bd \\ \hline ac - bc - ad + bd \\ \hline \end{array}$$

where $a - b$ is multiplied by c, and then by $-d$, and the two results collected into one.

On inspecting this example, it will readily appear that $+$ multiplied by $+$ gives $+$ in the product; $-$ multiplied by $-$ gives $+$; $+$ multiplied by $-$ gives $-$; and $-$ multiplied

by + gives −, or, in other words, *like* signs give +, and *unlike* signs give −.

Multiply $2x$ by 3

$$2x \times 3 = \text{three times } 2x = 2x + 2x + 2x = 6x,$$
$$\text{or thus, } 2 \times x \times 3 = 6x.$$

This result might have been obtained by multiplying the coefficient 2 by the multiolier 3, and writing the x after the product.

Multiply $5a$ by $2b$.

$$5 \times a \times 2 \times b = 5 \times 2 \times a \times b = 10ab.$$

This result might have been obtained by *multiplying the coefficients* 5 and 2 together, and writing the quantities a and b after the product.

Multiply a^2 by a^3.

$$a^2 = a \times a, \text{ and } a^3 = a \times a \times a;$$
$$\therefore a^2 \times a^3 = a \times a \times a \times a \times a = a^5;$$
$$\text{or thus, } a^2 \times a^3 = a^{2+3} = a^5.$$

This result might have been obtained by writing the quantity a with the *sum of the indices*, 2 and 3, for its index.

When no index is expressed, the index 1 is understood, thus, $a = a^1$.

From these considerations we derive the following

General Rules for Multiplication.

If the signs are *alike*, write +; if *unlike*, −. Multiply the coefficients. Add the indices of the same letter.

Examples.

1. Multiply $3a^2 - 5ab + 2b^2$ by $a^2 - 7ab$.

$$\begin{array}{rrrr}
3a^2 - & 5ab + & 2b^2 & \\
a^2 - & 7ab & & \\
\hline
3a^4 - & 5a^3b + & 2a^2b^2 & \\
 & -21a^3b + & 35a^2b^2 - & 14ab^3 \\
\hline
3a^4 - & 26a^3b + & 37a^2b^2 - & 14ab^3 \\
\hline
\end{array}$$

2. Multiply $a^n - a^{n-1}b + a^{n-2}b^n$ by $a - b$.

$$
\begin{array}{l}
a^n - a^{n-1}b^1 + a^{n-2}b^n \\
a^1 - b^1 \\
\hline
a^{n+1} - a^n b + a^{n-1}b^n \\
\qquad - a^n b \qquad\qquad + a^{n-1}b^2 - a^{n-2}b^{n+1} \\
\hline
a^{n+1} - 2a^n b + a^{n-1}b^n + a^{n-1}b^2 - a^{n-2}b^{n+1} \\
\hline
\end{array}
$$

3. Multiply $6a$ by 4; $3a$ by $5b$; and x^4 by x^2.
Ans. $24a$; $15ab$; and x^6.

4. Multiply $2x - 4y + z$ by $3x$; and $a^2 + 2ab - b^2$ by a^3b^2. Ans. $6x^2 - 12xy + 3xz$; and $a^5b^2 + 2a^4b^3 - a^3b^4$.

5. Multiply $x^3 - x^2y + xy^2 - y^3$ by $x + y$; and $a^2 + b^2 + c^2 + ab - ac + bc$ by $a - b + c$.
Ans. $x^4 - y^4$; and $a^3 - b^3 + c^3 + 3abc$.

6. Multiply $5x^3 + 4x^2 + 3x + 2$ by $5x^3 - 4x^2$; and $1 - x + x^4 - x^5$ by $1 + x + x^2 + x^3$.
Ans. $25x^6 - x^4 - 2x^3 - 8x^2$; and $1 - x^8$.

7. Find the product of $a + b$ and $a + b$; of $a - b$ and $a - b$; and of $a + b$ and $a - b$; and learn the results by heart.
Ans. $(a+b)(a+b) = a^2 + 2ab + b^2$; $(a-b)(a-b) = a^2 - 2ab + b^2$; and $(a+b)(a-b) = a^2 - b^2$.

From this example it appears that

1. The *square of the sum* of two quantities is equal to the *sum of their squares, together with twice their product.*

2. The *square of the difference* of two quantities is equal to the *sum of their squares, diminished by twice their product.*

3. The *product of the sum and difference* of two quantities is equal to the *difference of their squares.*

The student is recommended to commit these three theorems to memory.

DIVISION.

VII Division is just the reverse of Multiplication; thus,

$$\because 2x \times 3 = 6x; \quad \therefore 6x \div 3, \text{ or } \frac{6x}{3} = 2x;$$

$$\because\ -5x^4 \times 4x^2 = -20x^6,\quad \therefore\ \frac{-20x^6}{4x^2} = -5x^4$$

Hence, the *Rule for Division* will be; if the signs are *alike*, write $+$; if *unlike*, $-$; divide the coefficient of the dividend by the coefficient of the divisor; and subtract each index of the divisor from the index of the same letter in the dividend.

Examples

1. Divide $-18a^3bc^5$ by $-3a^2bc^3$.

$$\frac{-18a^3bc^5}{-3a^2bc^3} = 6ac^2.$$

Here, the signs being *alike*, the sign of the quotient is $+$ understood; the coefficient 3 is contained in 18, six times: the index 2 subtracted from 3 leaves 1 understood for the index of a in the quotient: b, being common to both dividend and divisor, is omitted in the quotient, since $\frac{b}{b} = 1$: and the index 3 subtracted from 5 leaves 2 for the index of c in the quotient.

2. Divide $12a^2x^4y^2 - 24ax^3y^4 - 18x^2y + 6xy$ by $-6xy$.

$$\frac{12a^2x^4y^2 - 24ax^3y^4 - 18x^2y + 6xy}{-6xy} = -2a^2x^3y + 4ax^2y^3 + 3x - 1$$

3. Divide $2a^7 - 7a^6b + 7a^5b^2 + a^4b^3 - 15a^3b^4$ by $2a^4 - 3a^3b - 5a^2b^2$.

$$\begin{array}{l}
2a^4 - 3a^3b - 5a^2b^2)\ 2a^7 - 7a^6b + 7a^5b^2 + a^4b^3 - 15a^3b^4\ (a^3 - 2a^2b + 3ab^2 \\
\qquad\qquad\qquad\qquad\ \underline{2a^7 - 3a^6b - 5a^5b^2} \\
\qquad\qquad\qquad\qquad\qquad -4a^6b + 12a^5b^2 + a^4b^3 \\
\qquad\qquad\qquad\qquad\qquad \underline{-4a^6b + 6a^5b^2 + 10a^4b^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 6a^5b^2 - 9a^4b^3 - 15a^3b^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{6a^5b^2 - 9a^4b^3 - 15a^3b^4}
\end{array}$$

4. Divide 1 by $1 + 2x + x^2$.

$$
\begin{array}{l}
1 + 2x + x^2)\ 1 \qquad (1 - 2x + 3x^2 - 4x^3 + \&c., \textit{ad inf.} \\
\qquad\qquad\quad \underline{1 + 2x + x^2} \\
\qquad\qquad\quad -2x - x^2 \\
\qquad\qquad\quad \underline{-2x - 4x^2 - 2x^3} \\
\qquad\qquad\qquad\quad 3x^2 + 2x^3 \\
\qquad\qquad\qquad\quad \underline{3x^2 + 6x^3 + 3x^4} \\
\qquad\qquad\qquad\qquad -4x^3 - 3x^4
\end{array}
$$

5. Divide $a^{2n+1} - a^{n+1} - a^n + a^{n-1}$ by a^{n-1}.

$$\frac{a^{2n+1} - a^{n+1} - a^n + a^{n-1}}{a^{n-1}} = a^{n+2} - a^2 - a + 1.$$

The indices in this quotient are obtained by Subtraction, thus;

$$
\begin{array}{cccc}
2n + 1 & n + 1 & n & n - 1 \\
\underline{n - 1} & \underline{n - 1} & \underline{n - 1} & \underline{n - 1} \\
n + 2 & 2 & 1 & 0
\end{array}
$$

It is obvious from this example that $a^0 = 1$:

$$\text{for } \frac{a^{n-1}}{a^{n-1}} = a^0:\ \text{but } \frac{a^{n-1}}{a^{n-1}} = 1 \quad \therefore a^0 = 1.$$

Examples, to be proved by Multiplication.

6. Divide $10a$ by 2: $-12ax$ by $4a$: $20x^4y^3$ by $-5xy^2$; and $-120a^6x^2y^5z^7$ by $-10a^4x^2y^2z^3$.

Ans. $5a$: $-3x$: $-4x^3y$; and $12a^2y^3z^4$.

7. Divide $3a^3b^2 - 6a^2b^4 + 12ab^6c$ by $3ab^2$; and $8x^4y^2 - 12x^2 - 16x$ by $4x$.

Ans. $a^2 - 2ab^2 + 4b^4c$; and $2x^3y^2 - 3x - 4$.

8. $\dfrac{2axy - 5xy}{xy}$; $\dfrac{4abx - 3a^2x + 7acx}{ax}$; and $\dfrac{-6b^2y + 4aby - 20by}{-by}$

Ans. $2a - 5$; $4b - 3a + 7c$; and $6b - 4a + 20$

9. Divide $8x^2 + 16ax + 6a^2$ by $4x + 2a$; and $x^4 + x^3y + xy^3 + y^4$ by $x^2 + 2xy + y^2$.

Ans. $2x + 3a$; and $x^2 - xy + y^2$

10. Divide $8x^4y + 2x^3y - 2x^2 - 3x^2y + x$ by $4x^2y + 3xy - 1$; and $x^4 - y^4$ by $x - y$.
Ans. $2x^2 - x$; and $x^3 + x^2y + xy^2 + y^3$.

11. Divide $18x^3 - 33x^2 + 44x - 35$ by $6x - 7$; and $1 - x$ by $1 + x$. Ans. $3x^2 - 2x + 5$; and $1 - 2x + 2x^2 - 2x^3 +$ &c.

12. $\dfrac{a^4 + 2a^2b^2 + b^4}{a^2 + b^2}$; $\dfrac{x^6 - 2x^3y^3 + y^6}{x^3 - y^3}$; $\dfrac{4a^8 - 1}{2a^4 + 1}$;

and $\dfrac{a^m . b^{2n+2} - a^{m+1} . b^{n+1} + a^{m+2} . b^{2n}}{a^{m-2}b^n}$

Ans. $a^2 + b^2$; $x^3 - y^3$; $2a^4 - 1$; and $a^2 b^{n+2} - a^3b^{n+1} + a^4b^n$.

Miscellaneous Examples.

1. Find the numerical values of the following expressions, when $x = 4$, $y = 3$, and $z = 6$.

(1.) $x^2 + 4y - 3z$. (2.) $\dfrac{2x + 6y}{13} - \dfrac{z}{y}$.

(3.) $\dfrac{x(z - y)}{3} + \dfrac{z}{x - y}$.

2. Add together $5a^2b - 2ab^2 - 3a^2b^2$, $2a^2b^2 - 7a^2b + 5ab^2$; and $ab^2 - a^2b$.

3. Subtract $2x^3 - 3x^2 + 8x - 5$ from $5x^3 - x^2 - 2x + 1$.

4. Multiply $4a^3x^2 - 7a^2x - 3a$ by $2ax^3 - a^2$; and $a^2 + ab + b^2$ by $a - b$.

5. Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$; and $x^3 + y^3$ by $x + y$; and prove each operation by multiplication.

6. Find the sum of $a\sqrt[3]{x} - b\sqrt{x}$, $5a\sqrt[3]{x} + 3b\sqrt{x}$, $2b\sqrt{x} - 7a\sqrt[3]{x}$, and $3a\sqrt[3]{x} + 4b\sqrt{x}$; and from $(3a - x)x$ take $(a + 2x)x$.

7. Separate into simple factors $a^2 - x^2$; $a^2 + 2x + x^2$; $x^4 - 1$; $4a^4 - 9b^2$; $x^2 - 4bx + 4b^2$; and $a^{2m} - b^{2n}$.

8. Divide $ax^3 - (a^2 + b)x^2 + b^2$ by $ax - b$; and multiply $px^2 + qx - r$ by $mx - n$.

9. Express in simple factors $a^2 a$; $mn - m$; $ab^3 - a$; $-bx^2 + bcx^2 - b^2x$; and $-c^2x - bx - x$.

10. Express without brackets $mn(m+n)$; $xy(x-y)$; $(p+r).(p-r)$; $-(r+av)x$; and $-\{x^2 + (x-1)\}y$.

SIMPLE EQUATIONS.

VIII. When two quantities, or combinations of quantities, are equal, they form an equation; thus $4 \times 3 = 7 + 5$ is an equation. Also $10x = 60$ is an equation, in which the value of x is evidently 6, since $10 \times 6 = 60$; and, if both sides of the equation be divided by 10, we shall have $x = 6$ at once. Again, $x + 5 = 15$ is an equation, from each side of which if we subtract 5, we shall have $x = 10$. Also $x - 4 = 20$ is an equation, to each side of which if we add 4, we shall have $x = 24$. And $\frac{1}{4}x = 2$ is an equation in which the value of x is 8, since the fourth part of 8 is 2; and, if both sides of this equation be multiplied by 4, we shall at once have $x = 8$.

Let $x + 2a = 8a - 2x$ be a given equation.

Add $2x$ to both sides, then

$$x + 2x + 2a = 8a.$$

Subtract $2a$ from both sides, then

$$x + 2x = 8a - 2a.$$

On comparing this with the original equation, it appears that the $2x$ and the $2a$ have each been *transposed*, or removed from one side of the equation to the other, and have had their *signs changed.*

$$\text{Now} \because x + 2x = 3x, \text{and } 8a - 2a = 6a$$
$$\therefore 3x = 6a$$

and taking a third part of each member of this equation, we have

$x = 2a$ for the value of the previously unknown quantity x.

Hence it appears that we may transpose any of the terms of an equation if we change their signs, and that whatever we do to one member of an equation, we must also do to the other, in order to preserve the equality.

Solve the following equations:

1. $x + 3 = 18 - 4x$. Ans. $x = 3$.
2. $x + 3a = 18a - 4x$. Ans. $x = 3a$.

3. $4x - 2a = 3x + 2b$. Ans. $x = 2(a + b)$.
4. $7 + 6x - 4 = 12 + 3x$. Ans. $x = 3$.
5. $4(x - 2) = 10x - 38$. Ans. $x = 5$.
6. $ax + c = a - bx$. Ans. $x = \frac{a - c}{a + b}$.
7. $\frac{y}{12} - 8 = -6$. Ans. $y = 24$.
8. $5az - 1 = 3a(z + b)$. Ans. $z = \frac{3ab + 1}{2a}$.

Problems.

1. A man buys a cow and her calf for $21l.$, and the cow is worth 6 times as much as the calf: what is the value of each?

Let x pounds represent the value of the calf;
then $\because$ the cow is worth 6 times as much,
$\therefore 6x$ is the value of the cow.

But, by the question, the value of the cow + the value of the calf = 21 pounds.

$$\therefore 6x + x = 21$$
that is, $7x = 21$
$$\therefore x = 3 \text{ pounds} = \text{value of calf},$$
$$\text{and } 6x = 18 \text{ pounds} = \ldots\ldots \text{ cow}.$$

2. A had $100l.$ and B $48l.$; B gave a certain sum away, and A twice as much, and then A had three times as much as B had. What did A give away?

Let x = number of pounds B gave away,
then $2x$ = ... A ......

Now each man's whole money minus the money he gave away = the money he had left.

$\therefore 100 - 2x$ = money A had left,
and $48 - x$ = ... B ......

But, by the question, A had left 3 times as much as B,

$$\therefore 100 - 2x = 3(48 - x)$$
that is, $100 - 2x = 144 - 3x$,
By transposition, $3x - 2x = 144 - 100$;
that is, $x = 44l.$ = B's gift,
and $2x = 88l.$ = A's gift.

3. Divide 600*l.* among three persons, A, B, and C, so that B may have twice as much as A, and C as much as A and B together. Ans. A 100*l.*, B 200*l.*, and C 300*l.*

4. The difference of two numbers is 7, and if 8 times the less number be subtracted from 3 times the greater, the remainder will be 6: find the numbers. Ans. 10 and 3.

5. A man and his son earn 96 shillings in a month: now the man's labour is 5 times as valuable as the son's: how much ought each to receive? Ans. The man 4*l.*, the son 16*s.*

6. Two persons, A and B, invest equal sums of money in railway shares; A gains 300*l.* and B loses 450*l.*, after which A's money is 6 times as much as B's: what money did each invest? Ans. 600*l.*

7. A line, a feet long, is to be divided into two such parts that one part may be b times the length of the other: find the length of each part

Ans. $\frac{a}{b+1}$ feet = less part, $\frac{ab}{b+1}$ feet = greater.

Find the length of the parts when $a = 20$ and $b = 4$.
Ans. 4 and 16.

CHAPTER II.

FRACTIONS.

IX. Algebraic Fractions are the same in principle as Fractions in Common Arithmetic, consequently the same rules apply to both. Suppose any unit, as an orange, to be divided into b equal parts, and a of those parts to be taken, we shall then have the fraction $\frac{a}{b}$, just as, when we divide the unit into 8 equal parts and take 5 of them, we have the fraction $\frac{5}{8}$.

Since $\frac{a}{b}$ expresses the *quotient* of a by b, and since quotient × divisor = dividend

$$\therefore \frac{a}{b} \times b = a;$$

that is, if we *multiply a fraction by its denominator*, we *obtain its numerator*.

Multiply each member of the equation by n, then

$$nb \times \frac{a}{b} = na$$

$$\therefore \frac{a}{b} = \frac{na}{nb};$$

that is, the value of a fraction is not altered by multiplying both numerator and denominator by the same quantity.

Cor. $\because \frac{na}{nb} = \frac{a}{b}$, $\therefore$ the value of a fraction is not altered by dividing both numerator and denominator by the same quantity.

Examples.

1. Multiply $\frac{x}{2}$ by 2; $\frac{x}{3}$ by 12; $\frac{2y}{5}$ by 15; and $\frac{3y}{4}$ by $5x$.

$\frac{x}{2} \times 2 = x$, $\because$ twice the half of anything = the whole of that thing.

$\frac{x}{3} \times 12 = 4x$, $\because$ 12 times the third of anything = 12 thirds = 4 times the whole of that thing. Hence, when we have to multiply a fraction by a number, we may, if we can, previously divide the multiplier by the denominator.

$\frac{2y}{5} \times 15 = 6y$, $\because$ 15 times $\frac{2}{5} = \frac{30}{5} = 6$, or $\because 15 \div 5 = 3$, and $2y \times 3 = 6y$.

$\frac{3y}{4} \times 5x = \frac{15xy}{4}$ $\because$ 5 times $\frac{3}{4} = \frac{15}{4}$, and $y \times x = xy$.

2. Multiply $\frac{x}{3} + \frac{3x}{4} - \frac{5x}{6} + \frac{3x}{8} - \frac{7x}{12}$ by 24, and add the results. Ans. $8x + 18x - 20x + 9x - 14x = x$.

3. Multiply $\frac{3y}{2} - \frac{2y}{5} - \frac{y}{4} + \frac{7y}{10} - \frac{5y}{4}$ by 20, and add the results. Ans. $6y$.

4. Multiply $\frac{4}{x} - \frac{7}{2x} + \frac{3}{8x} - \frac{9}{4x} + \frac{3}{2}$ by $8x$, and add the results. Ans. $12x - 11$.

5. Multiply $\frac{x}{5}+\frac{5}{x}-\frac{x}{10}-\frac{2}{x^2}+3$ by $10\,x^2$.

Ans. $x^3+30x^2+50x-20$.

These examples will aid the student in clearing equations from fractions.

GREATEST COMMON MEASURE.

X. A *measure* of a quantity is any factor which will divide it without remainder: thus 2 is a measure of $2\,a$; ax is a measure of $a\,x^2$; $a+b$ is a measure of $(a+b)\,a$.

A *common measure* of two or more quantities is any factor which will divide them all without remainder; and their *greatest common measure* is the greatest factor which will so divide them: thus a is a common measure of $2\,a$, $6\,a^2x$, and $4\,a\,x^2$, and their greatest common measure is $2\,a$: also $3\,(a+b)$ is the greatest common measure of $6\,(a+b)^2$, $9\,(a^2-b^2)$ and $12\,(a^3-b^3)$, for $6\,(a+b)^2=6\,(a+b)\,.\,(a+b)$, $9\,(a^2-b^2)=9\,(a+b)(a-b)$, and $12\,(a+b^3)=12\,(a+b)(a^2-ab+b^2)$ where $3\,(a+b)$ is evidently the greatest common factor.

This mode of proceeding, by separating the quantities into their simple factors, will frequently enable the student to ascertain the greatest common measure.

Examples.

1. Find the G. C. M. of x^2+2x-3 and $x^2+5\,x+6$.

$x^2+2\,x-3=x^2+3\,x-x-3=(x+3)\,x-(x+3)$
$=(x+3)\,(x-1)$

$x^2+5\,x+6=x^2+3x+2\,x+6=x\,(x+3)+2\,(x+3)$
$=(x+3)\,(x+2)$

$\therefore\ x+3$ is the G. C. M.

If $x^2+2\,x-3$ and $x^2+5\,x+6$ were the two terms of a fraction, it could be reduced to its lowest terms by dividing both numerator and denominator by the greatest common measure $x+3$; thus

$\frac{x^2+2\,x-3}{x^2+5\,x+6}=\frac{(x+3)\,(x-1)}{(x+3)\,(x+2)}=\frac{x-1}{x+2}$, the fraction in its lowest terms.

2. Find the G. C. M. of $\frac{x^3-6x^2+11x-6}{x^3+4x^2+x-6}$, and reduce the fraction to its lowest terms.

$$\frac{x^3-6x^2+11x-6}{x^3+4x^2+x-6}=\frac{x^3-x^2-5x^2+5x+6x-6}{x^3-x^2+5x^2-5x+6x-6}$$

$$=\frac{x^2(x-1)-5x(x-1)+6(x-1)}{x^2(x-1)+5x(x-1)+6(x-1)}$$

$$=\frac{(x-1)(x^2-5x+6)}{(x-1)(x^2+5x+6)}=\frac{x^2-5x+6}{x^2+5x+6}$$

= the fraction in its lowest terms, and $x-1=$ G. C. M

3. $$\frac{x^2-9x+20}{x^2+6x-55}=\frac{x^2-4x-5x+20}{x^2-5x+11x-55}$$

$$=\frac{x(x-4)-5(x-4)}{x(x-5)+11(x-5)}=\frac{(x-5)(x-4)}{(x-5)(x+11)}$$

$$=\frac{x-4}{x+11}=\text{the fraction in its lowest terms,}$$

and $x-5=$ G. C. M.

4. $$\frac{x^8+x^6y^2+x^2y+y^3}{x^4-y^4}=\frac{x^6(x^2+y^2)+y(x^2+y^2)}{(x^2+y^2)(x^2-y^2)}$$

$$=\frac{x^6+y}{x^2-y^2}=\text{the fraction in its lowest terms,}$$

and $x^2+y^2=$ G. C. M.

5. $$\frac{ab+2a^2-3b^2-4bc-ac-c^2}{9ac+2a^2-5ab+4c^2+8bc-12b^2}$$

$$=\frac{2a^2+ac+3ab-2ab-bc-3b^2-2ac-c^2-3bc}{2a^2-8ab+8ac+ac-4bc+4c^2+3ab-12b^2+12bc}$$

$$=\frac{a(2a+c+3b)-b(2a+c+3b)-c(2a+c+3b)}{2a(a-4b+4c)+c(a-4b+4c)+3b(a-4b+4c)}$$

$$=\frac{(a-b-c)(2a+c+3b)}{(a-4b+4c)(2a+c+3b)}=\frac{a-b-c}{a-4b+4c}$$

= the fraction in its lowest terms,

and $2a+c+3b=$ G. C. M.

XI. A quantity is a *multiple* of another when it contains that other a certain number of times without remainder: thus $6a$ is a multiple of $2a$; nx is a multiple of x.

If a quantity measures another, it will also measure any multiple of that other. Thus, suppose b measures a by the units in m, that is, let b be contained m times in a, then $a = mb$, and let na be a multiple of a, then $na = nmb$; hence b measures na by the units in nm.

If a quantity measures two others, it will also measure their sum and difference. Thus, let x measure a by the units in m, and b by the units in n, then $a = mx$, and $b = nx$; $\therefore a \pm b = mx \pm nx = (m \pm n)\,x$: hence x measures their sum $(a + b)$ by the units in $(m + n)$, and their difference $(a - b)$ by the units in $(m - n)$.

We can now investigate a general rule for finding the greatest common measure of any two numbers.

Let the numbers be represented by a and b, a being $> b$.

Suppose b is contained in a, p times with remainder c, then $a = pb + c$;

Suppose c is contained in b, q times with remainder d, then $b = qc + d$;

Suppose d is contained in c, r times with remainder 0, then $c = rd$.

$$\begin{array}{l}
b)\,a\,(p \\
\quad pb \\
\quad \overline{\quad c)\,b\,(q} \\
\qquad\quad qc \\
\qquad\quad \overline{\quad d)\,c\,(r} \\
\qquad\qquad\quad dr \\
\qquad\qquad\quad \overline{\quad 0}
\end{array}$$

Now $\because d$ measures c, $\therefore d$ measures qc, and $qc + d$, that is, b;

$\therefore d$ measures pb and $pb + c$, that is a.

Hence the last divisor d measures both the given numbers a and b.

d is the greatest common measure of a and b: for suppose δ to be the greatest common measure, then

$\because \delta$ measures a and b $\therefore$ δ measures a and p , and $a - pb$, that is, c;

$\therefore$ δ measures qc, and $b - qc$, that is, d.

Hence $\because \delta$, the greatest common measure of a and b, measures d;

and $\because d$ is a common measure of a and b;

and also $\because$ the greatest number that measures d must be equal to d;

$\therefore \delta = d$;

$\therefore$ the last divisor d is the greatest common measure of a and b.

Hence the rule may be thus expressed: "Divide the greater number by the less, and that divisor by the remainder, and so on till nothing remains; the last divisor will be the greatest common measure."

This rule would apply to the examples in Art. X.

LEAST COMMON MULTIPLE

A *common multiple* of any quantities is a quantity which contains them all without remainder, and the *least common multiple* is the least quantity which so contains them.

Let m be any common measure of two quantities a and b,

$$\text{let } \frac{a}{m} = p, \frac{b}{m} = q; \text{ then by multiplication}$$

$$\frac{ab}{m^2} = pq, \frac{ab}{m} = mpq = aq \text{ or } bp,$$

$$\therefore \frac{ab}{m} \text{ is a common multiple of } a \text{ and } b.$$

But when m is greatest $\frac{ab}{m}$ is least; that is

$$\frac{ab}{\text{G.C.M.}} = \text{the L. C. M. of } a \text{ and } b.$$

Hence, to find the *least common multiple* of two quantities, we may divide their *product* by their *greatest common measure*.

A similar proof will apply to any number of quantities.

The L.C.M. may generally be found by inspection.

1. Find the G. C. M. of $\dfrac{3x^3 - 3x^2y + xy^2 - y^3}{4x^2 - xy - 3y^2}$, and reduce the fraction to its lowest terms.

Here since $4x^2$ is not contained in $3x^3$, we multiply the

numerator by 4 before commencing the division; the factor 4 being introduced into only *one* of the terms, will evidently not affect the greatest *common* measure.

$$
\begin{array}{l}
4x^2 - xy - 3y^2)\,12x^3 - 12x^2y + 4xy^2 - 4y^3(3x - 9y \\
12x^3 - 3x^2y - 9xy^2 \\
\hline
 - 9x^2y + 13xy^2 - 4y^3 \text{ Multiply by 4.} \\
 - 36x^2y + 52xy^2 - 16y^3 \\
 - 36x^2y + 9xy^2 + 27y^3 \\
\hline
 43xy^2 - 43y^3
\end{array}
$$

Reject the factor $43y^2$.

$$
\begin{array}{l}
x - y)\,4x^2 - xy - 3y^2(4x + 3y \\
4x^2 - 4xy \\
\hline
 3xy - 3y^2 \\
 3xy - 3y^2 \\
\hline
\end{array}
$$

Hence $x - y$ is the G. C. M., and dividing both terms of the given fraction by it, we have

$$\frac{3x^3 - 3x^2y + xy^2 - y^3}{4x^2 - xy - 3y^2} = \frac{3x^2 + y^2}{4x + 3y} \quad \text{Ans.}$$

2. Show that $\dfrac{x^6 + x^{-2} - (x^{-6} + x^2)}{x^3 + x - (x^{-3} + x^{-1})} = \dfrac{x^4 - x^{-4}}{x + x^{-1}}$.

$$\frac{x^6 - x^{-2} - (x^{-6} + x^2)}{x^3 + x - (x^{-3} + x^{-1})} = \frac{x^{-2}(x^8 + 1) - x^{-6}(1 + x^8)}{x(x^2 + 1) - x^{-3}(1 + x^2)} =$$

$$\frac{(x^{-2} - x^{-6})(x^8 + 1)}{(x - x^{-3})(x^2 + 1)} = \frac{x^{-6}(x^4 - 1)(x^8 + 1)}{x^{-3}(x^4 - 1)(x^2 + 1)} = \frac{x^{-4}(x^8 + 1)}{x^{-1}(x^2 + 1)} =$$

$$\frac{x^4 + x^{-4}}{x + x^{-1}} \quad \text{Ans.}$$

Find the G. C. M. of

1. $x^2 + x - 2$ and $x^2 + 2x - 3$. Ans. $x - 1$.
2. $6a^2 + 11ax + 3x^2$ and $6a^2 + 7ax - 3x^2$. Ans. $2a + 3x$.
3. $8a^2b^2 - 10ab^3 + 2b^4$ and $9a^4b - 9a^3b^2 + 3a^2b^3 - 3ab^4$. Ans. $ab - b^2$.
4. $x^3 + 2x^2 + 2x + 1$ and $x^3 - 2x - 1$. Ans. $x + 1$

5. $2x^2 - xy - 6y^2$, and $3x^2 - 8xy + 4y^2$. Ans. $x - 2y$.

6. $2x^3 - 3x^2 - 2x + 3$, and $3x^4 + 2x^3 - 2x^2 - 2x - 1$. Ans. $x^2 - 1$

Reduce to the lowest terms

7. $\dfrac{x^2+2ax+a^2}{x^2-a^2}$ and $\dfrac{x^2-2xy+y^2}{x^2-y^2}$. Ans. $\dfrac{x+a}{x-a}$ and $\dfrac{x-y}{x+y}$.

8. $\dfrac{a^3+b^3}{(a+b)^2}$ and $\dfrac{(a+b)^3}{a^2-b^2}$. Ans. $\dfrac{a^2-ab+b^2}{a+b}$ and $\dfrac{(a+b)^2}{a-b}$.

9. $\dfrac{x^2+x-2}{2x^2-3x+1}$ and $\dfrac{x^{3m}+x^{2m}-2}{x^{2m}+x^m-2}$.

Ans. $\dfrac{x+2}{2x-1}$ and $\dfrac{x^{2m}+2x^m+2}{x^m+2}$.

10. $\dfrac{x^3-x^2y-xy^2+y^3}{x^4-y^4}$ and $\dfrac{3x^3-22x-15}{5x^4-17x^3+18x}$.

Ans $\dfrac{x-y}{x^2+y^2}$ and $\dfrac{3x^2+9x+5}{5x^3-2x^2-6x}$

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF ALGEBRAIC FRACTIONS.

Examples.

1. Add together $\dfrac{2x-5}{3}$ and $\dfrac{x-1}{2x}$.

Here the least common multiple of the denominators is $6x$, and the two fractions can be reduced so as to have a common denominator by multiplying both terms of the first by $2x$, and both terms of the second by 3; thus,

$$\left.\begin{aligned}\frac{2x-5}{3}\times\frac{2x}{2x}&=\frac{4x^2-10x}{6x}\\ \frac{x-1}{2x}\times\frac{3}{3}&=\frac{3x-3}{6x}\end{aligned}\right\}\ \therefore\ \frac{4x^2-7x-3}{6x}=\text{the sum.}$$

2. Find the sum of $\frac{1}{2}a - \frac{1}{3}b + \frac{1}{5}c$, $\frac{1}{4}a - \frac{1}{5}b - \frac{1}{8}c$ and $\frac{1}{3}a + \frac{1}{4}b + \frac{1}{2}c$.

$$\frac{1}{2}a - \frac{1}{3}b + \frac{1}{5}c = \frac{6}{12}a - \frac{20}{60}b + \frac{6}{30}c$$

$$\frac{1}{4}a - \frac{1}{5}b - \frac{1}{3}c = \frac{3}{12}a - \frac{12}{60}b - \frac{10}{30}c$$

$$\frac{1}{3}a + \frac{1}{4}b + \frac{1}{2}c = \frac{4}{12}a + \frac{15}{60}b + \frac{15}{30}c$$

$$\therefore \frac{13}{12}a - \frac{17}{60}b + \frac{11}{30}c = \text{sum required}$$

3. Of $\frac{5x^2}{2} - \frac{7y^2}{3}$, $-\frac{4x^2}{3} + \frac{5y^2}{2}$ and $\frac{7x^2}{4} - \frac{3y^2}{2}$.

$$\frac{5x^2}{2} - \frac{7y^2}{3} = \quad \frac{30x^2}{12} - \frac{14y^2}{6}$$

$$-\frac{4x^2}{3} + \frac{5y^2}{2} = -\frac{16x^2}{12} + \frac{15y^2}{6}$$

$$\frac{7x^2}{4} - \frac{3y^2}{2} = \quad \frac{21x^2}{12} - \frac{9y^2}{6}$$

$$\therefore \frac{35}{12}x^2 - \frac{8}{6}y^2 = 2\tfrac{11}{12}x^2 - 1\tfrac{1}{3}y^2 = \text{sum.}$$

4. Collect $\frac{3}{4(1-x)^2} + \frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x)^2}$ into a single fraction.

Taking the second and third, and reducing them to a common denominator,

$$\left.\begin{aligned} \frac{3}{8(1-x)} \times \frac{1+x}{1+x} &= \frac{3+3x}{8(1-x^2)} \\ \frac{1}{8(1+x)} \times \frac{1-x}{1-x} &= \frac{1-x}{8(1-x^2)} \end{aligned}\right\} = \frac{4+2x}{8(1-x^2)} = \frac{2+x}{4(1-x^2)} \text{ by}$$

addition.

Taking the fraction just found and the fourth, and reducing them to a common denominator,

$$\left.\begin{array}{l}\dfrac{2+x}{4(1-x^2)}\times\dfrac{1+x^2}{1+x^2}=\dfrac{x^3+2x^2+x+2}{4(1-x^4)}\\[2ex] \dfrac{1-x}{4(1+x)}\times\dfrac{1-x^2}{1-x^2}=\dfrac{x^3-x^2-x+1}{4(1-x^4)}\end{array}\right\}=\frac{3x^2+2x+1}{4(1-x^4)}\text{ by}$$

subtraction.

Taking the fraction last found and the first, and reducing them to a common denominator, first finding the least common multiple, that is, dividing the product of the denominators by their G. C. M., $4(1-x)$,

$$4(1-x)\overline{)\,4(1-x)^2\,4(1-x^2)(1+x^2)}$$

$$(1-x)4(1-x^2)(1+x^2)=4(1-x)(1-x^4)=\text{L C.M.}$$

$$\left.\begin{array}{l}\dfrac{3}{4(1-x)^2}\times\dfrac{1+x+x^2+x^3}{1+x+x^2+x^3}=\dfrac{3+3x+3x^2+3x^3}{4(1-x)(1-x^4)}\\[2ex] \dfrac{1+2x+3x^2}{4(1-x^4)}\times\dfrac{1-x}{1-x}=\dfrac{1+x+x^2-3x^3}{4(1-x)(1-x^4)}\end{array}\right\}=$$

$$\frac{4+4x+4x^2}{4(1-x)(1-x^4)}=\frac{1+x-x^2}{1-x-x^4+x}\text{ by addition.}$$

5. Find the difference between $a+b$ and $\frac{a}{2}-\frac{b}{2}$.

$$a+b=\frac{2a}{2}+\frac{2b}{2}$$

$$\frac{a}{2}-\frac{b}{2}$$

$$\therefore\ \frac{a}{2}+\frac{3b}{2}=\text{difference required.}$$

6. Find the difference between $\frac{a}{3}+\frac{b}{2}-\frac{c}{4}$ and $\frac{a+b}{4}-\frac{c}{8}$

$$\frac{a}{3}+\frac{b}{2}-\frac{c}{4}=\frac{4a}{12}+\frac{6b}{12}-\frac{2c}{8}$$

$$\frac{a+b}{4}-\frac{c}{8}=\frac{3a}{12}+\frac{3b}{12}-\frac{c}{8}$$

$$\frac{a}{12}+\frac{3b}{12}-\frac{c}{8}=\frac{a+3b}{12}-\frac{c}{8}.\quad\text{Ans.}$$

7. Find the sum and difference of $\frac{x}{x-2}$ and $\frac{x}{x-3}$.

$$\frac{x}{x-2} \times \frac{x-3}{x-3} = \frac{x^2-3x}{x^2-5x+6}$$

$$\frac{x}{x-3} \times \frac{x-2}{x-2} = \frac{x^2-2x}{x^2-5x+6}$$

$\therefore$ by addition $\frac{2x^2-5x}{x^2-5x+6}$ = the sum,

and by subtraction $\frac{-x}{x^2-5x+6}$ = the difference.

8. Multiply $\frac{4x^2-1}{8}$ by $\frac{2x+1}{2x-1}$; and $\frac{2a^2}{a^2-b^2}$ by $\frac{(a+b)^2}{4b^2a^2}$.

$$\frac{4x^2-1}{8} \times \frac{2x+1}{2x-1} = \frac{(2x+1)(2x-1)}{8} \times \frac{2x+1}{2x-1}$$

$$= \frac{(2x+1)(2x+1)}{8} = \frac{4x^2+4x+1}{8}.$$

$$\frac{2a^2}{a^2-b^2} \times \frac{(a+b)^2}{4b^2a^2} = \frac{2a^2}{(a+b)(a-b)} \times \frac{(a+b)(a+b)}{2b^2 \times 2a^2}$$

$$= \frac{a+b}{2b^2(a-b)}.$$

9. Divide $\frac{a+1}{b}$ by $\frac{1+\frac{1}{a}}{b}$; and $\left(\frac{1}{a}+\frac{1}{b}\right)$ by $\left(\frac{1}{a}-\frac{1}{b}\right)$.

$$\frac{a+1}{b} \div \frac{1+\frac{1}{a}}{b} = \frac{a+1}{b} \div \frac{{}^*a+1}{ab} = \frac{a+1}{b} \times \frac{ab}{a+1} = a$$

$$\left(\frac{1}{a}+\frac{1}{b}\right) \div \left(\frac{1}{a}-\frac{1}{b}\right) = \frac{b+a}{ab} \div \frac{b-a}{ab} =$$

$$\frac{b+a}{ab} \times \frac{ab}{b-a} = \frac{b+a}{b-a}.$$

* This is obtained by multiplying both terms of the divisor by a.

10. Multiply $\dfrac{x^2-(a+b)x+ab}{x^2-(a-b)x-ab}$ by $\dfrac{x+b}{x-b}$.

$$\frac{x^2-ax-bx+ab}{x^2-ax+bx-ab}=\frac{x^2-bx-ax+ab}{x^2+bx-ax-ab}=$$

$$\frac{x(x-b)-a(x-b)}{x(x+b)-a(x+b)}=\frac{(x-a)(x-b)}{(x-a)(x+b)}=\frac{x-b}{x+b}$$

$$\text{and } \frac{x-b}{x+b}\times\frac{x+b}{x-b}=1.$$

11. Collect into one expression $\dfrac{a^m}{(a+b)^n}+\dfrac{a^{m-2}b^2}{(a+b)^{n-1}}$ $-\dfrac{a^{m-3}b^2}{(a+b)^{n-2}}$.

Here $\because$ $(a+b)^n$ is the L. C. M. of the denominators, we take the second and third fractions, and reduce them to the denominator $(a+b)^n$; thus

$$\left.\begin{aligned}\frac{a^{m-2}b^2}{(a+b)^{n-1}}\times\frac{a+b}{a+b}&=\frac{a^{m-1}b^2+a^{m-2}b^3}{(a+b)^n}\\ \frac{a^{m-3}b^2}{(a+b)^{n-2}}\times\frac{(a+b)^2}{(a+b)^2}&=\frac{a^{m-1}b^2+2a^{m-2}b^3+a^{m-3}b^4}{(a+b)^n}\end{aligned}\right\}$$

$$=\frac{-a^{m-2}b^3-a^{m-3}b^4}{(a+b)^n}\text{ by subtraction,}$$

and prefixing the first fraction, we have

$$\frac{a^m-(a^{m-2}+a^{m-3}b)b^3}{(a+b)^n}=\text{the expression required.}$$

Examples in Addition of Fractions.

1. Add together $\dfrac{x}{2}$, $\dfrac{2x}{3}$ and $\dfrac{3x}{4}$. Ans. $\dfrac{23x}{12}=x+\dfrac{11}{12}x$.

2. $\dfrac{2x}{5}+\dfrac{3x}{2}+\dfrac{7x}{10}$; and $\dfrac{x}{a+x}+\dfrac{a}{a-x}$.

Ans. $\dfrac{13x}{5}$; and $\dfrac{a^2+2ax-x^2}{a^2-x^2}$.

3. $\frac{3a^2}{2b} + \frac{2a}{5} + \frac{3b}{7a}$; and $\frac{x}{x+3} + \frac{x}{x-3}$.

Ans. $\frac{105a^3 + 28a^2b + 30b^2}{70ab}$; and $\frac{2x^2}{x^2-9}$.

4. $\frac{a+b}{a-b} + \frac{a-b}{a+b}$; and $\frac{2x}{1-x^2} + \frac{1}{x+1}$.

Ans. $\frac{2(a^2+b^2)}{a^2-b^2}$; and $\frac{1}{1-x}$.

5. $\frac{a^2+ab+b^2}{a+b} + \frac{b^2}{a-b}$; and $\frac{x-1}{x^2+x+1} + \frac{1}{x-1}$.

Ans. $\frac{a(a^2+b^2)}{a^2-b^2}$; and $\frac{2x^2-x+2}{x^3-1}$.

6 $\frac{x}{x^2-y^2} + \frac{y}{x+y} + \frac{1}{x-y}$; and $\frac{a}{1-a} + \frac{1}{1+a} + \frac{a}{1+a}$

Ans. $\frac{2x+xy-y^2+y}{x^2-y^2}$; and $\frac{1}{1-a}$.

Examples in Subtraction of Fractions.

1. From $\frac{5a}{2}$ take $\frac{a}{3}$; and from $\frac{3(x+y)}{4}$ take $\frac{x-y}{8}$.

Ans. $\frac{13}{6}a$; and $\frac{5x+7y}{8}$.

2. $\frac{5y+2}{7} - \frac{2y+1}{3}$; and $\frac{5x-3}{x+1} - \frac{3x+2}{x-1}$.

Ans. $\frac{y-1}{21}$; and $\frac{2x^2-13x+1}{x^2-1}$.

3 $\frac{a+2b}{a-2b} - \frac{a-2b}{a+2b}$; and $\frac{1}{x-y} - \frac{1}{x+y}$.

Ans $\frac{8ab}{a^2-4b^2}$; and $\frac{2y}{x^2-y^2}$.

4. $\frac{1}{y-z} - \frac{1}{y^2-z^2}$; and $\frac{2x^2-2x+1}{x^2-x} - \frac{x}{x-1}$.

Ans. $\frac{y+z-1}{y^2-z^2}$; and $1-\frac{1}{x}$.

5. $\dfrac{x^2-x+1}{x-1}-\dfrac{2}{x+1}$; and $\dfrac{a}{(1-a)^2}-\dfrac{a^2}{(1-a)^3}+\dfrac{1}{1-a}$.

Ans. $\dfrac{x^3-2x+3}{x^2-1}$; and $\dfrac{1-a-a^2}{(1-a)^3}$.

6. $\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x}{x+y}-\dfrac{y}{x-y}$; and $1-\dfrac{1}{(x-y)^2}-\dfrac{1}{x^2-y^2}$.

Ans. $\dfrac{2x}{x+y}$; and $1-\dfrac{2y}{(x+y)(x-y)^2}$.

7 $\dfrac{1}{2}\cdot\dfrac{3m+2n}{3m-2n}-\dfrac{1}{2}\cdot\dfrac{3m-2n}{3m+2n}$. Ans. $\dfrac{12mn}{9m^2-4n^2}$.

8. $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$. Ans. $x^{2n}+2$.

Examples in Multiplication of Fractions.

1. Multiply $\dfrac{3a}{5}$ by $\dfrac{a}{4}$; and $\dfrac{2x}{3}$ by $\dfrac{xy^2}{6}$.

Ans. $\dfrac{3a^2}{20}$; and $\dfrac{x^2y^2}{9}$.

2. $\dfrac{2x}{x-y}\times\dfrac{x^2-y^2}{8}$; and $\dfrac{x^2-1}{3}\times\dfrac{6a}{x+1}$.

Ans. $\dfrac{x^2+xy}{4}$; and $2a(x-1)$.

3. $\dfrac{a^2-b^2}{a}\times\dfrac{1}{a+b}\times\dfrac{a}{a-b}$; and $\dfrac{15z-30}{2z}\times\dfrac{3z^2}{5z-10}$.

Ans. 1; and $\dfrac{9}{2}z$.

4. $\dfrac{(x-1)^2}{y^3}\times\dfrac{(x+1)y^3}{x-1}$; and $\left(m+\dfrac{1}{m}-1\right)\times$ $\left(m+\dfrac{1}{m}+1\right)$. Ans. $\dfrac{x^2-1}{y}$; and $m^2+1+\dfrac{1}{m^2}$.

5 $\left(x-\dfrac{y^2}{x}\right)\left(\dfrac{x}{y}+\dfrac{y}{x}\right)$; and $\dfrac{4x^4-1}{9x^6-y^2}\cdot\dfrac{3x^3+y}{2x^2-1}$.

Ans. $\dfrac{x^4-y^4}{x^2y}$; and $\dfrac{2x^2+1}{3x^3-y}$.

6. $\left\{\frac{a^4 x}{a^2 - x^2} - a^2 x - x^3\right\} \times \frac{a + x}{x^4}$. Ans. $\frac{x^2}{a - x}$.

Examples in Division of Fractions.

1. Divide $\frac{5m^2}{n}$ by $\frac{10}{3n}$; and $\frac{3x}{2x - 2}$ by $\frac{2x}{x - 1}$.

Ans. $\frac{3}{2}m^2$; and $\frac{3}{4}$.

2. $\frac{4a + 2}{3} \div \frac{2a + 1}{5a}$; and $\frac{(x + y)^2}{x - y} \div \frac{x + y}{(x - y)^2}$.

Ans. $\frac{10a}{3}$; and $x^2 - y^2$.

3. $\left(x + \frac{x}{x - 1}\right) \div \left(x - \frac{x}{x - 1}\right)$; and $\left(x^2 - \frac{1}{x^2}\right) \div \left(x^2 + \frac{1}{x}\right)$. Ans. $\frac{x}{x - 2}$; and $x^2 - \frac{1}{x}$.

4. $\frac{x^3 - 3x^2 a + 3xa^2 - a^3}{x + a} \div \frac{x - a}{x + a}$. Ans. $(x - a)^2$.

5. $\frac{x^4 - y^4}{a^3 + b^3} \div \frac{x - y}{a^2 - ab + b^2}$. Ans. $\frac{x^3 + x^2 y + xy^2 + y^3}{a + b}$.

6. $\left(x^2 + 2 + \frac{1}{x^2}\right) \div \frac{x + \frac{1}{x}}{a}$ Ans. $\frac{ax^2 + a}{x}$

Find the value of x in the following equations.

1. $3x + 4 = \frac{5x + 4}{2} + 8$. Multiply by 2.

$6x + 8 = 5x + 4 + 16$ Transpose.
$6x - 5x = 4 + 16 - 8$
$\therefore x = 12$.

2. $\frac{30 + x}{x} - 5 = \frac{6}{x}$. Multiply by x.

$30 + x - 5x = 6$. Transpose.
$x - 5x = 6 - 30$
$-4x = -24$. Divide by -4.
$\therefore x = 6$.

3. $\frac{x}{2}-\frac{*5x+4}{3}=7-\frac{*8x-2}{3}$. Multiply by 6, the L.C.M. of the denominators

$3x-10x-8=42-16x+4$. Transpose.

$$3x-10x+16x=42+4+8$$
$$9x=54$$
$$\therefore x=6.$$

4. $\frac{2}{x+1}+\frac{5}{2x+2}+\frac{6x-6}{x^2-1}=2\frac{5}{8}$.

Here $\frac{6x-6}{x^2-1}=\frac{6(x-1)}{(x+1)(x-1)}$, hence the equation becomes

$\frac{2}{x+1}+\frac{5}{2(x+1)}+\frac{6}{x+1}=\frac{21}{8}$. Multiply by $8(x+1)$

$$16+20+48=21(x+1)$$
$$84=21(x+1)$$
$$4=x+1$$
$$\therefore 3=x.$$

5. $\frac{ax+b}{c}+\frac{ax+b}{cx+b}=\frac{2ax+d}{2c}+\frac{b}{c}$. Multiply by $2c$.

$$2ax+2b+\frac{2acx+2bc}{cx+b}=2ax+d+2b.$$

$\because$ $2ax$ and $2b$ are found on each side of the equation with the same sign, they may be omitted; and the equation becomes

$\frac{2acx+2bc}{cx+b}=d$. Multiply by $cx+b$.

$2acx+2bc=cdx+bd$. Transpose.

$$2acx-cdx=bd-2bc$$
$$(2ac-cd)x=bd-2bc$$
$$\therefore x=\frac{bd-2bc}{2ac-cd}.$$

* The sign $-$ in front of these fractions changes the signs of their numerators; for it indicates that each fraction is to be *subtracted* from the quantity preceding it.

6. $$\frac{7x+\frac{13}{2}}{10}+\frac{11x-\frac{x-\frac{3}{2}}{2}}{12}=\frac{3x+1}{5}+\frac{43x-\frac{3-8x}{2}}{22}.$$

Here $\dfrac{7x+\frac{13}{2}}{10}=\dfrac{14x+13}{20}$.

$$\frac{11x-\frac{x-\frac{3}{2}}{2}}{12}=\frac{11x-\frac{2x-3}{4}}{12}=\frac{44x-2x+3}{48}=$$
$$\frac{42x+3}{48}=\frac{14x+1}{16}.$$

$$\frac{43x-\frac{3-8x}{2}}{22}=\frac{86x-3+8x}{44}=\frac{94x-3}{44}.$$

Hence the equation becomes

$$\frac{14x+13}{20}+\frac{14x+1}{16}=\frac{3x+1}{5}+\frac{94x-3}{44}.$$

Multiply by 880, the L. C. M. of the denominators.

$$616x+572+770x+55=528x+176+1880x-60,$$
$$1386x-2408x=116-627$$
$$-1022x=-511$$
$$\therefore\ x=\frac{-511}{-1022}=\frac{1}{2}.$$

Problems.

1. Divide 100 into two such parts that, if the one be divided by 15 and the other by 5, the sum of the quotients shall be 10.

Let x be one part, then $100-x$ will be the other.

Now $\dfrac{x}{15}$ and $\dfrac{100-x}{5}$ are the two quotients.

Hence, by the question, $\dfrac{x}{15}+\dfrac{100-x}{5}=10$.

$\therefore$ Multiplying each side by 15,

$$x + 300 - 3x = 150$$
$$-2x = 150 - 300$$
$$= -150$$
$$\therefore x = 75 = \text{one part,}$$
$$\text{and } 100 - x = 100 - 75 = 25 = \text{the other}$$

Or thus; let x and y be the parts,

$$\text{then } x + y = 100 \quad (1)$$
$$\text{and } \frac{x}{15} + \frac{y}{5} = 10 \quad (2). \quad \text{Multiply by 15.}$$

$$\left.\begin{array}{rr} & x + 3y = 150 \\ (1) & x + y = 100 \end{array}\right\} \text{Subtract.}$$

$$2y = 50 \therefore y = 25 = \text{one part.}$$
$$(1) \quad x = 100 - y = 100 - 25 = 75 = \text{the other.}$$

2. A person in a foreign town wishes to exchange a sovereign for 25 pieces of the two kinds of coin used there; and he finds that 30 of the one, or 15 of the other, are equal to a sovereign. How many must he have of each?

Let x be the number of one kind,

and y .... the other;

then $x + y = 25$ .... (1)

$$\begin{array}{ccccccc} \text{coins.} & & \text{coins.} & & \text{shill.} & & \text{shill.} \\ 30 & : & x & :: & 20 & : & \frac{2x}{3} = \text{value of the } x \text{ coins.} \\ 15 & : & y & :: & 20 & : & \frac{4y}{3} = \text{..............} y \text{...} \end{array}$$

Hence, by the question $\frac{2x}{3} + \frac{4y}{3} = 20$ shillings ...... (2)

$$\left.\begin{array}{ll} \text{Multiply (2) by 3,} & 2x + 4y = 60 \\ \text{Multiply (1) by 2,} & 2x + 2y = 50 \end{array}\right\} \text{Subtract}$$

$$2y = 10 \;..\; y = 5$$
$$(1) \quad x = 25 - y = 25 - 5 = 20$$

Ans. 20 and 5

3. A person lays by $\frac{1}{4}$th of what he spends. His income being diminished, while his expenditure remains the same, he finds he now lays by only half the former sum. What part of the original income was the diminution?

Let $2x$ pounds be what he lays by,
then $8x$ ...... is .... spends;

$\therefore\ 2x + 8x = 10x =$ his original income.

Again $8x$ is still what he spends,
x .... lays by;

$\therefore\ 8x + x = 9x =$ his diminished income,

$\therefore\ 10x - 9x = x =$ the diminution,

$$\text{Hence } \frac{x}{10x} = \frac{1}{10} = \text{Ans.}$$

4. A performs a journey at a certain rate; had he travelled $\frac{1}{2}$ a mile an hour quicker, he would have performed the journey in $\frac{4}{5}$ of the time; but had he travelled $\frac{1}{2}$ a mile slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance and his rate.

Let x miles be his rate per hour,
and y miles be the whole distance,

then $\frac{y}{x}$ is the number of hours the journey occupies.

Now $x + \frac{1}{2}$ or $\frac{2x+1}{2}$ is the increased rate;

$\therefore\ y \div \frac{2x+1}{2}$ or $\frac{2y}{2x+1}$ is the hours he would be on the road, by travelling $\frac{1}{2}$ a mile an hour quicker.

$$\text{Hence, by the question, } \frac{2y}{2x+1} = \frac{4}{5} \cdot \frac{y}{x} \text{ .... (1)}$$

Again, $x - \frac{1}{2}$ or $\frac{2x-1}{2}$ is the diminished rate;

$\therefore\ y \div \frac{2x-1}{2}$ or $\frac{2y}{2x-1}$ is the hours he would be on the road, by travelling $\frac{1}{2}$ a mile an hour slower.

Hence, by the question, $\dfrac{2y}{2x-1} = \dfrac{y}{x} + 2\frac{1}{2}$ .... (2)

Divide (1) by $2y$, $\dfrac{1}{2x+1} = \dfrac{2}{5x}$

$$5x = 4x + 2$$
$$5x - 4x = 2 \quad \therefore x = 2$$

Substitute this value of x in (2)

$$\frac{2y}{4-1} = \frac{y}{2} + \frac{5}{2}$$

$$\frac{2y}{3} = \frac{y}{2} + \frac{5}{2}. \quad \text{Multiply by 6.}$$

$$4y = 3y + 15$$
$$4y - 3y = 15 \qquad \therefore y = 15$$

Ans. Distance = 15 miles; rate = 2 miles per hour.

5. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but 2 of the greyhound's leaps are as much as 3 of the hare's; how many leaps must the greyhound take to catch the hare?

Let x = the number of leaps.

$2 : x :: 3 : \dfrac{3x}{2} =$ { The number of leaps the hare takes from the time she starts till overtaken.

$3 : x :: 4 : \dfrac{4x}{3} =$ { The number of leaps the hare takes from the time the greyhound starts.

$$\text{Hence } \frac{3x}{2} - 50 = \frac{4x}{3}$$

$$9x - 300 = 8x$$
$$x = 300. \quad \text{Ans.}$$

6. A person in play lost a fourth of his money, and then won back 3*s*., after which he lost a third of what he had left, and then won back 2*s*.; lastly, he lost a seventh of what he then had, and after this found he had but 12*s*. remaining: what had he at first?

Let $x =$ the money he had at first in shillings,

then $\frac{x}{4}$ he lost,

$x - \frac{x}{4} = \frac{3x}{4}$ he had left,

$\therefore \frac{3x}{4} + 3$ or $\frac{3x + 12}{4}$ he had after he won 3*s*.,

$\frac{x + 4}{4}$ he lost the second time,

$\therefore \frac{3x + 12}{4} - \frac{x + 4}{4} = \frac{2x + 8}{4} = \frac{x + 4}{2}$ he had left,

$\frac{x + 4}{2} + 2$ or $\frac{x + 8}{2}$ he had after he won 2*s*,

hence $\frac{x + 8}{14}$ he lost the third time,

$\frac{x + 8}{2} - \frac{x + 8}{14}$ or $\frac{7x + 56}{14} - \frac{x + 8}{14} = \frac{6x + 48}{14}$ he had left,

hence $\frac{6x + 48}{14} = 12$ by the question,

or $\frac{x + 8}{14} = 2$

$x + 8 = 28$

$\therefore x = 28 - 8 = 20s.$ Ans.

7. A company of foot march 1165 of their own paces ahead of a troop of horse; now. if the foot take 5 paces to every 4 of the horse, but 3 paces of a horse be equal in extent to 4 paces of the foot, how many paces will the horse have marched before they overtake the foot?

Let x be the number of steps the horse must take,

	horse steps		horse steps		foot steps		foot steps
then	3	:	x	::	4	:	$\frac{4x}{3}$

= number of footsteps the infantry march from the time they start till overtaken.

$$\begin{array}{ccccccc} \text{horse steps} & & \text{horse steps} & & \text{foot steps} & & \text{foot steps} \\ 4 & : & x & :: & 5 & : & \dfrac{5x}{4} \end{array}$$

= number of footsteps the infantry march from the time the cavalry start.

$$\therefore \frac{4x}{3} - 1165 = \frac{5x}{4}$$

$$16x - 13980 = 15x$$

$$16x - 15x = 13980$$

$$\therefore x = 13980.$$

8. A waterman finds by experience that he can, with the advantage of a common tide, row down a river from A to B, which is 18 miles, in an hour and a half; and that to return from B to A against an equal tide, though he rows back along the shore, where the stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter. It is required from hence to find at what rate per hour the tide runs in the middle, where it is strongest.

Let x = the rate per hour the tide runs in the middle,

then $\frac{3x}{5}$ = ... at the side.

And $\frac{18}{1\frac{1}{2}} = 12$ miles = the rate per hour he can row in the middle, with the advantage of the tide;

$\therefore 12 - x$ = the rate per hour he can row without the tide.

Again, $\frac{18}{2\frac{1}{4}} = 8$ miles = the rate per hour he can row against the tide at the side;

$\therefore 8 + \frac{3x}{5}$ = the rate he can row without any tide

$$\text{Hence } 12 - x = 8 + \frac{3x}{5}$$

$$60 - 5x = 40 + 3x$$

$$8x = 20$$

$x = 2\frac{1}{2}$ miles = the rate per hour the tide runs in the middle, where it is strongest.

Exercises in Simple Equations.

1. $\frac{5x}{6} + \frac{2}{3} = x + \frac{1}{3}$. $\qquad x = 2$.

2. $\frac{3x - 4}{2} = \frac{x}{2} + \frac{x}{4} + 1$ $\qquad x = 4$.

3. $5x - \frac{6x + 3}{11} = \frac{7x + 15}{2} - 3$. $\qquad x = 5$.

4. $\frac{4}{x} + 2 = \frac{24}{x} - \frac{1}{2}$. $\qquad x = 8$.

5. $\frac{x}{8} - 1 + \frac{x}{12} - \frac{x + 5}{4} = -\frac{11}{4}$. $\qquad x = 12$.

6. $\frac{x + a}{b} - \frac{x}{a} = 1$. $\qquad x = -a$.

7. $\frac{b}{ax} - a^2 = b^2 - \frac{a}{bx}$. $\qquad x = \frac{1}{ab}$.

8. $\frac{a(b^2 + x^2)}{bx} = ac + \frac{a}{b}x$. $\qquad x = \frac{b}{c}$.

Problems.

1. Ten years ago a boy's age was $\frac{1}{10}$ of his father's, but now it is $\frac{1}{4}$ of it; what are both their ages? Ans. 15 and 60.

2 If from three times a certain number we subtract 8, half the remainder will be equal to the number itself diminished by 2; what is the number? Ans. 4.

3 If to the numerator of a certain fraction we add one, its value will be $\frac{1}{3}$, but if from its denominator we subtract one, its value will be $\frac{1}{4}$; what is the fraction? Ans. $\frac{2}{9}$.

4. £132 is to be divided between A, B, and C, so that B will receive $\frac{4}{5}$ as much as A, and C $\frac{5}{6}$ as much as A and B together; find the share of each.

Ans. A £40, B £32, C £60.

5. A, having a certain number of sovereigns in his purse, and meeting two of his creditors, B and C, gave to B $\frac{3}{10}$ of the money, and to C $\frac{6}{7}$ of the remainder; he then found that he had left exactly eleven pounds; what money had he at first? Ans. £110.

6. A grazier spent $\frac{1}{12}$ of his money in horses, $\frac{1}{3}$ in oxen, and $\frac{3}{10}$ of the remainder in sheep; after which he found he had £98 left; how much had he at first? Ans. £240.

CHAPTER III

INVOLUTION

XII. Involution is the raising of a quantity to any required *power* by multiplication.

Thus, $a \times a = a^1 \times a^1 = a^{1+1} = a^2 =$ the 2nd power, or *square* of a.

$$a \times a \times a = a^{1+1+1} = a^3 = \text{the 3rd power, or cube of } a.$$

$$a \times a \times a \times a \ldots \text{to } n \text{ terms} = a^{1+1+1+1+\text{ \&c., to } n \text{ terms}} = a^n.$$

Again, $a^2 \times a^2 \times a^2 = a^{2+2+2} = a^{2\times 3} = (a^2)^3 = a^6$.

$$a^n \times a^n \times a^n \ldots \text{to } m \text{ terms} = a^{n+n+n\ldots\text{to } m \text{ terms}} = a^{n\times m} = a^{nm}.$$

Hence, to find any power of a quantity, we multiply the index of that quantity by the number indicating the power to which it is to be raised.

$$\sqrt{a} \times \sqrt{a} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a,$$

$$\sqrt[3]{a} \times \sqrt[3]{a} = a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}} = a^{\frac{2}{3}} = \sqrt[3]{a^2},$$

$$\sqrt[4]{a} \times \sqrt[4]{a} \times \sqrt[4]{a} = a^{\frac{1}{4}} \times a^{\frac{1}{4}} \times a^{\frac{1}{4}} = a^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = a^{\frac{3}{4}} = \sqrt[4]{a^3}.$$

$$(-a)^2 = -a \times -a = a^2,$$

$$(-a)^3 = -a \times -a \times -a = -a^3,$$

$$(-a)^4 = -a \times -a \times -a \times -a = a^4,$$

&c. &c. &c.

Hence, *even* powers of negative quantities are *positive*; and *odd* powers of negative quantities are *negative*.

$$(a+b)^2 = (a+b) \times (a+b) = a^2 + 2ab + b^2$$

$$(a-b)^2 = (a-b) \times (a-b) = a^2 - 2ab + b^2$$

$$(a+b)^3 = (a+b) \times (a+b) \times (a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$
$$= a^3 + b^3 + 3ab(a+b)$$

$$(a-b)^3 = (a-b) \times (a-b) \times (a-b) = a^3 - 3a^2b + 3ab^2 - b^3$$
$$= a^3 - b^3 - 3ab(a-b).$$

Hence, by the last two formulæ, the cube of the sum of two quantities is equal to the sum of their cubes + three times their product multiplied by their sum.

The cube of the difference of two quantities is equal to the difference of their cubes — three times their product multiplied by their difference.

$$(a+b+c)^2 = \{(a+b)+c\}^2 = (a+b)^2 + 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2$$

$$(a+b+c+d)^2 = \{(a+b)+(c+d)\}^2 =$$
$$(a+b)^2 + 2(a+b)(c+d)(c+d)^2.$$

$$(a+b-c+d)^2 = \{(a+b)-(c-d)\}^2 =$$
$$(a+b)^2 - 2(a+b)(c-d) + (c-d)^2.$$

Examples.

1. Show that $(a-2x)^2 = a^2 - 4ax + 4x^2$
2. $\left(x+\frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$
3. $\left(x+\frac{p}{2}\right)^2 = x^2 + px + \frac{p^2}{4}.$
4. $(2x-3y)^2 = 4x^2 - 12xy + 9y^2.$
5. $(2x-1)^2 = 4x^2 - 4x + 1.$
6. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$
7. $(a-x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$
8. $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$
9. $(s-z)^5 = s^5 - 5s^4z + 10s^3z^2 - 10s^2z^3 + 5sz^4 - z^5.$
10. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2 = x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = x + 2\sqrt{xy} + y.$
11. $(x - \sqrt{y})^2 = x^2 - 2x\sqrt{y} + y.$
12. $(x^2y + \sqrt{y})^2 = x^4y^2 + 2x^2y^{\frac{3}{2}} + y.$
13. $(x^{\frac{1}{3}} - y^{\frac{1}{3}})^3 = x - 3x^{\frac{2}{3}}y^{\frac{1}{3}} + 3x^{\frac{1}{3}}y^{\frac{2}{3}} - y.$
14. $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.$
15. $\{(a+b)^{\frac{1}{3}} - (a-b)^{\frac{1}{3}}\}^3 =$
$$2b - 3(a+b)^{\frac{1}{3}}(a-b)^{\frac{1}{3}}\{(a+b)^{\frac{1}{3}} - (a-b)^{\frac{1}{3}}\}.$$

EVOLUTION.

XIII. EVOLUTION is the extraction of roots.

The square of a being a^2, the square root of a^2 is a.

The cube of a being a^3, the cube root of a^3 is a.

The square root of a is $\sqrt{a}$, or $a^{\frac{1}{2}}$; $\because$ $\sqrt{a} \times \sqrt{a} = a$, and $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{2}{2}} = a^1 = a$.

The cube root of a is $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$.

The nth root of a is $\sqrt[n]{a}$, or $a^{\frac{1}{n}}$.

The cube root of a^2 is $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$

The mth root of a^n is $\sqrt[m]{a^n}$, or $a^{\frac{n}{m}}$.

Hence, to find any root of a quantity, we divide the index by the number indicating the root required;

$$\because \; +a \times +a = +a^2, \text{ and } -a \times -a = +a^2$$

$$\therefore \; \sqrt{a^2} = +a, \text{ or } -a.$$

Hence the square root of a quantity is either positive, or negative.

$$\because \; +a \times +a \times +a = +a^3, \text{ but} -a \times -a \times -a = -a^3;$$

$$\therefore \; \sqrt[3]{a^3} = +a, \text{ but } \sqrt[3]{-a^3} = -a.$$

Hence the cube root of a positive quantity is positive, but the cube root of a negative quantity is negative.

Since there is no quantity, which, being multiplied by itself, will produce $-a$, $\therefore$ $\sqrt{-a}$ is an *impossible* quantity.

$a^2 + 2ab + b^2$ being the square of $a + b$, we may, by means of this formula, investigate a rule for extracting the square root of a binomial; for $a^2 + 2ab + b^2 = a^2 + (2a + b)b$, and $\because$ the first term of the root is a, the square root of a^2, we have only to find a divisor which will give the quotient $+b$, and this divisor is evidently $2a + b$; that is, twice the first term of the root together with the second term of the root.

Hence, to find the square root of $a^2 + 2ab + b^2$, we first take a, the square root of a^2, and, placing it as we should a quotient, we subtract its square from $a^2 + 2ab + b^2$; we then double the root a for a divisor, and divide the remaining portion of the quantity, namely $(2a + b)\,b$, by $2a$, and annex the quotient b to the a, and also to the divisor; we then multiply $2a + b$ by this second term b and subtract the result.

The process may be thus exhibited,

$$\begin{array}{rl}
 & a^2 + 2ab + b^2\,(a + b \\
 & a^2 \\
\hline
2a + b) & 2ab + b^2 \\
 & 2ab + b^2 \\
\hline
\end{array}$$

Let us apply this rule to the extraction of the square root of the number 1225.

$$\begin{array}{rl}
 & 1225\ (30 + 5,\ \text{or}\ 35 \\
 & 900^{*} \\
\hline
60 + 5,\ \text{or}\ 65) & 325 \\
 & 325 \\
\hline
\end{array}$$

Let it be required to extract the square root of $a^2 + 2ab + 2ac + 2bc + b^2 + c^2$.

$$\begin{array}{rl}
 & a^2 + 2ab + 2ac + 2bc + b^2 + c^2\,(a + b + c \\
 & a^2 \\
\hline
2a + b) & 2ab \qquad\qquad + b^2 \\
 & 2ab \qquad\qquad + b^2 \\
\hline
2a + 2b + c) & 2ac + 2bc \quad + c^2 \\
 & 2ac + 2bc \quad + c^2 \\
\hline
\end{array}$$

In this example, having obtained $a + b$, this portion of the root is doubled for a new divisor, in order to obtain $+ c$, the remaining portion of the root.

Since the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$, we may readily derive a rule for the extraction of the cube root. The cube root of the first term a^3, is a, the first term of the root; we subtract its cube from the whole quantity; then, for a divisor we take $3a^2$, and dividing the first term of the remainder by it, we obtain b, the second term of the root; then annexing $3ab + b^2$ to the divisor, and multiplying it by b, we obtain $3a^2b + 3ab^2 + b^3$, which is equal to the remainder.

* In the ordinary arithmetical process these two ciphers would be omitted.

The process is as follows,

$$\begin{array}{ll}
 & a^3 + 3a^2b + 3ab^2 + b^3\,(a + b \\
 & a^3 \\
\hline
3a^2 + 3ab + b^2) & \quad 3a^2b + 3ab^2 + b^3 \\
 & \quad 3a^2b + 3ab^2 + b^3 \\
\hline
\end{array}$$

We will apply this rule to the extraction of the cube root of 91125.

$$\begin{array}{rr}
 & 91125\ (\overset{a}{40} + \overset{b}{5},\ \text{or } 45 \\
a^3 = & 64000 \\
\hline
3a^2 = 3 \times 40^2 = 4800) & 27125 \\
\hline
3a^2b = 3 \times 40^2 \times 5 = & 24000 \\
3ab^2 = 3 \times 40 \times 5^2 = & 3000 \\
b^3 = 5^3 \qquad\qquad = & 125 \\
\hline
 & 27125 \\
\hline
\end{array}$$

The ordinary arithmetical process would be

$$\begin{array}{rr}
 & 9\dot{1}12\dot{5}\ (45 \\
 & 64 \\
\hline
4^2 \times 3 = 48) & 27125 \\
\hline
5^3 \qquad = & 125 \\
5^2 \times 4 \times 3 = & 300 \\
5 \times 48 \qquad = & 240 \\
\hline
 & 27125 \\
\hline
\end{array}$$

Examples.

1. Find the square root of $4a^2 + 4ax + x^2$.

$$\begin{array}{ll}
 & 4a^2 + 4ax + x^2\ (2a + x = \text{the root.} \\
 & 4a^2 \\
\hline
4a + x) & \quad 4ax + x^2 \\
 & \quad 4ax + x^2 \\
\hline
\end{array}$$

2. $\sqrt{(29x^2y^2 + 30xy^3 + 12x^3y + 25y^4 + 4x^4)}$.

$$
\begin{array}{lll}
 & 9x^2y^2 + 20x^2y^2 + 30xy^3 + 12x^3y + 25y^4 + 4x^4 & \\
 & 9x^2y^2 & (3xy + 2x^2 + 5y^2. \\
\hline
6xy + 2x^2) & 12x^3y \quad + 4x^4 & \\
 & 12x^3y \quad + 4x^4 & \\
\hline
6xy + 4x^2 + 5y^2) & 30xy^3 + 20x^2y^2 + 25y^4 & \\
 & 30xy^3 + 20x^2y^2 + 25y^4 & \\
\hline
\end{array}
$$

In this example $29x^2y^2 = 9x^2y^2 + 20x^2y^2$, and the other terms are brought down in the order which is found to be most convenient.

3. $\sqrt{(a^2 - x^2)}$

$$
\begin{array}{ll}
 & a^2 - x^2 \left(a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \text{\&c.} = \text{the root.} \right. \\
 & a^2 \\
\hline
2a - \frac{x^2}{2a}) & - x^2 \\
 & - x^2 + \frac{x^4}{4a^2} \\
\hline
2a - \frac{x^2}{a} - \frac{x^4}{8a^3}) & - \frac{x^4}{4a^2} \\
 & - \frac{x^4}{4a^2} + \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\
\hline
2a - \frac{x^2}{a} - \frac{x^4}{4a^3}) & - \frac{x^6}{8a^4} - \frac{x^8}{64a^6}.
\end{array}
$$

Here the process does not terminate, and the root is merely an approximate value of $\sqrt{(a^2 - x^2)}$.

4. Extract the cube root of $8a^3 - 84a^2x + 294ax^2 - 343x^3$.

$$
\begin{array}{rl}
 & 8a^3 - 84a^2x + 294ax^2 - 343x^3\ (2a - 7x \\
 & 8a^3 \\
 & \overline{} \\
(2a)^2 \times 3 = 12a^2) & -84a^2x + 294ax^2 - 343x^3 \\
(-7x) \times (2a)^2 \times 3 = & -84a^2x \\
(-7x)^2 \times 2a \times 3 = & \qquad\qquad + 294ax^2 \\
(-7x)^3 \qquad = & \qquad\qquad\qquad\qquad - 343x^3 \\
 & \overline{-84a^2x + 294ax^2 - 343x^3.}
\end{array}
$$

5. $\sqrt[3]{}(a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3).$

$$
\begin{array}{rl}
 & a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3\ (a+b+c. \\
 & a^3 \\
3a^2) & 3a^2b \\
b \times a^2 \times 3 = & 3a^2b \\
b^2 \times a \times 3 = & \qquad 3ab^2 \\
b^3 \qquad = & \qquad\qquad + b^3 \\
(a+b)^2 \times 3 = 3a^2 + 6ab + 3b^2) & 3a^2c \quad +6abc+3ac^2 \quad +3b^2c+3bc^2+c^3 \\
c \times (a+b)^2 \times 3 = & 3a^2c \quad +6abc \quad +3b^2c \\
c^2 \times (a+b) \times 3 = & \qquad +3ac^2 \quad +3bc^2 \\
c^3 \qquad = & \qquad\qquad +c^3
\end{array}
$$

6. The square root of $4x^2 - 4ax + a^2 = 2x - a$.

7. $\sqrt{(1-8a+16a^2)} = 1-4a$; and $\sqrt{(25a^2+40ax+16x^2)} = 5a + 4x$.

8. $\sqrt{\left(9x^2 - 3x + \frac{1}{4}\right)} = 3x - \frac{1}{2}$.

9. $\sqrt{(49a^4 + 42a^2b + 9b^2)} = 7a^2 + 3b$.

10. $\sqrt{(x^2 - 2ax + a^2 + 2x - 2a + 1)} = x - a + 1$.

11. $\sqrt{\left(m^2 + 2m - 1 - \frac{2}{m} + \frac{1}{m^2}\right)} = m + 1 - \frac{1}{m}$.

12. $\sqrt{\left(\frac{a^2}{x^2} - 2 + \frac{x^2}{a^2} + \frac{2a^2}{x} - 2x + a^2\right)} = \frac{a}{x} - \frac{x}{a} + a$.

13. $\sqrt[3]{(a^3 - 3a^2x + 3ax^2 - x^3)} = a - x$.

14. $\sqrt[3]{(1 - 12a + 48a^2 - 64a^3)} = 1 - 4a$.

15. $\sqrt[3]{(8x^3 + 36ax^2 + 54a^2x + 27a^3)} = 2x + 3a$.

16. $\sqrt[3]{(x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1)} = x^2 - 2x + 1$.

17. $\sqrt[3]{(y^6 - 6y^5 + 6y^4 + 16y^3 - 12y^2 - 24y - 8)} = y^2 - 2y - 2$

CHAPTER IV.

SURDS.

XIV SURDS, or Irrational quantities, are such that the roots indicated cannot be exactly determined; thus,

$\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{4}$, $\sqrt[3]{3^2}$, $\sqrt{(1+x^2)}$, $\sqrt[3]{(x^2+ax)}$, and $\sqrt[3]{2x^2}$ are surds.

These quantities might be expressed by using fractional indices, thus, $2^{\frac{1}{2}}$, $5^{\frac{1}{2}}$, $4^{\frac{1}{3}}$, $3^{\frac{2}{3}}$, $(1+x^2)^{\frac{1}{2}}$, $(x^2+ax)^{\frac{1}{3}}$, and $(2x^2)^{\frac{1}{3}}$.

Any rational quantity may be made to assume the form of a surd; thus, $x = \sqrt{x^2} = x^{\frac{2}{2}}$, $x - a = \sqrt[n]{(x-a)^n} = (x-a)^{\frac{n}{n}}$, $a\sqrt{x} = a^{\frac{2}{2}}x^{\frac{1}{2}} = (a^2x)^{\frac{1}{2}} = \sqrt{a^2x}$.

Examples.

1. Prove that $\sqrt{243} + \sqrt{27} + \sqrt{48} = 16\sqrt{3}$.

$$\sqrt{243} = \sqrt{81 \times 3} = 9\sqrt{3}$$
$$\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$$
$$\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$$

$$\therefore \sqrt{243} + \sqrt{27} + \sqrt{48} = 9\sqrt{3} + 3\sqrt{3} + 4\sqrt{3}$$
$$= (9 + 3 + 4)\sqrt{3} = 16\sqrt{3}$$

2. Show that $2\sqrt{8a^3} - 7a\sqrt{18a} + 5\sqrt{72a^3} - \sqrt{50ab^2}$ $= (13a - 5b)\sqrt{2a}$.

$$2\sqrt{8a^3} = 2\sqrt{4a^2 \times 2a} = 2 \times 2a\sqrt{2a} = 4a\sqrt{2a}$$
$$-7a\sqrt{18a} = -7a\sqrt{9 \times 2a} = -7a \times 3\sqrt{2a} = -21a\sqrt{2a}$$
$$5\sqrt{72a^3} = 5\sqrt{36a^2 \times 2a} = 5 \times 6a\sqrt{2a} = 30a\sqrt{2a}$$
$$-\sqrt{50ab^2} = -\sqrt{25b^2 \times 2a} = -5b\sqrt{2a};$$

and $4a\sqrt{2a} - 21a\sqrt{2a} + 30a\sqrt{2a} - 5b\sqrt{2a} =$
$$(4a - 21a + 30a - 5b)\sqrt{2a} = (13a - 5b)\sqrt{2a}.$$

3. Multiply
$$\left(4\sqrt{\tfrac{7}{3}} + 5\sqrt{\tfrac{1}{2}}\right) \text{by} \left(\sqrt{\tfrac{7}{3}} + 2\sqrt{\tfrac{1}{2}}\right).$$

This operation can be performed in the same way as the multiplication of ordinary algebraic quantities; thus,

$$4\sqrt{\frac{7}{3}} + 5\sqrt{\frac{1}{2}}$$
$$\sqrt{\frac{7}{3}} + 2\sqrt{\frac{1}{2}}$$
$$\overline{4 \times \frac{7}{3} + 5\sqrt{\frac{7}{6}}}$$
$$8\sqrt{\frac{7}{6}} + 10 \times \frac{1}{2}$$
$$\overline{\frac{28}{3} + 13\sqrt{\frac{7}{6}} + 5}$$
$$= \frac{28}{3} + \frac{15}{3} + 13\sqrt{\frac{7}{6}} = \frac{43}{3} + 13\sqrt{\frac{7}{6}}.$$

4. Prove that $12\sqrt[3]{\frac{1}{4}} + 3\sqrt[3]{\frac{1}{32}} = \frac{27}{4}\sqrt[3]{2}$

$$12\sqrt[3]{\frac{1}{4}} = 12\sqrt[3]{\frac{2}{8}} = \frac{12}{2}\sqrt[3]{2}$$

$$3\sqrt[3]{\frac{1}{32}} = 3\sqrt[3]{\frac{2}{64}} = \frac{3}{4}\sqrt[3]{2},$$

$$\text{and } \frac{12}{2}\sqrt[3]{2} + \frac{3}{4}\sqrt[3]{2} = \frac{24}{4}\sqrt[3]{2} + \frac{3}{4}\sqrt[3]{2} = \frac{27}{4}\sqrt[3]{2}.$$

5. Reduce $(2\sqrt{8} + 3\sqrt{5} - 7\sqrt{2})$ and
$(\sqrt{72} - 5\sqrt{20} - 2\sqrt{2})$ to their simplest form, and find their product.

$$2\sqrt{8} = 2\sqrt{4 \times 2} = 2 \times 2\sqrt{2} = 4\sqrt{2}$$

$$\therefore 2\sqrt{8} + 3\sqrt{5} - 7\sqrt{2} = 4\sqrt{2} + 3\sqrt{5} - 7\sqrt{2}$$
$$= 3\sqrt{5} - 3\sqrt{2}$$

$$\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2};$$

$$\text{and } -5\sqrt{20} = -5\sqrt{4 \times 5} = -10\sqrt{5}$$

$$\therefore \sqrt{72} - 5\sqrt{20} - 2\sqrt{2} = 6\sqrt{2} - 10\sqrt{5}$$
$$- 2\sqrt{2} = 4\sqrt{2} - 10\sqrt{5};$$

$$\begin{array}{l} 3\sqrt{5} - 3\sqrt{2} \\ 4\sqrt{2} - 10\sqrt{5} \\ \hline 12\sqrt{10} - 12 \times 2 \\ \qquad - 30 \times 5 + 30\sqrt{10} \\ \hline 42\sqrt{10} - 24 - 150 = 42\sqrt{10} - 174 \end{array}$$

6. Express as a single fraction

$$\frac{a + b\sqrt{-1}}{a - b\sqrt{-1}} + \frac{a - b\sqrt{-1}}{a + b\sqrt{-1}}.$$

Reducing these fractions to a common denominator,

$$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}\times\frac{a+b\sqrt{-1}}{a+b\sqrt{-1}}=\frac{a^2+2ab(-1)+b^2(-1)}{a^2-b^2(-1)}$$

$$=\frac{a^2-2ab-b^2}{a^2+b^2};$$

$$\frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}\times\frac{a-b\sqrt{-1}}{a-b\sqrt{-1}}=\frac{a^2-2ab(-1)+b^2(-1)}{a^2-b^2(-1)}$$

$$=\frac{a^2+2ab-b^2}{a^2+b^2}.$$

Hence by addition,

$$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}+\frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}=\frac{2a^2-2b^2}{a^2+b^2}=\frac{2(a^2-b^2)}{a^2+b^2}.$$

This example exhibits a method of making a binomial surd rational; for the denominators of the given fractions are each multiplied by a quantity which has made them rational. In general since $(x+y)(x-y)=x^2-y^2$, any surd of the form $a\pm\sqrt{b}$ or $\sqrt{a}\pm\sqrt{b}$ may be made rational by multiplying by a similar surd of the form $a\mp\sqrt{b}$ or $\sqrt{a}\mp\sqrt{b}$ using the upper sign when the *sum* is given, and the lower when the *difference* is given.

7. Show that

$$\frac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}=\frac{\sqrt{30}+3\sqrt{2}+2\sqrt{3}}{12}.$$

Multiplying both terms of the given fraction by

$\sqrt{2}+\sqrt{3}+\sqrt{5}$, we have

$$\frac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}\times\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$$

$$=\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2\sqrt{6}}$$

Now, multiplying both terms by $\sqrt{6}$, we have

$$\frac{\sqrt{12}+\sqrt{18}+\sqrt{30}}{2\times 6}=\frac{2\sqrt{3}+3\sqrt{2}+\sqrt{30}}{12}.$$

8. Multiply $a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

$$
\begin{array}{l}
a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}} \\
a^{\frac{1}{4}} - b^{\frac{1}{2}} \\
\hline
a + a^{\frac{3}{4}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b + a^{\frac{1}{4}}b^{\frac{3}{2}} \\
\quad - a^{\frac{3}{4}}b^{\frac{1}{2}} - a^{\frac{1}{2}}b - a^{\frac{1}{4}}b^{\frac{3}{2}} - b^2 \\
\hline
a \quad * \quad * \quad * \quad - b^2.
\end{array}
$$

9. Show that

$$\sqrt{\left\{1 + \frac{11a}{4} - \sqrt{3a}\,(1 + a) + a^2\right\}} = 1 - \frac{\sqrt{3}}{2}\sqrt{a} + a.$$

Extract the square root of the expression within the brackets.

$$
\begin{array}{lll}
 & 1 + \frac{11}{4}a - \sqrt{3}.a^{\frac{1}{2}} - \sqrt{3}.a^{\frac{3}{2}} + a^2 & \left(1 - \frac{\sqrt{3}}{2}a^{\frac{1}{2}} + a.\right. \\
 & 1 & \\
2 - \frac{\sqrt{3}}{2}a^{\frac{1}{2}}\Big) & - \sqrt{3}.a^{\frac{1}{2}} + \frac{11}{4}a & \\
 & - \sqrt{3}.a^{\frac{1}{2}} + \frac{3}{4}a & \\
\hline
2 - \sqrt{3}.a^{\frac{1}{2}} + a\Big) & 2a - \sqrt{3}.a^{\frac{3}{2}} + a^2 & \\
 & 2a - \sqrt{3}.a^{\frac{3}{2}} + a^2 & \\
\hline
 & * \quad * \quad * &
\end{array}
$$

10. Show that $\sqrt{12} + \sqrt{27} - \sqrt{3} + \sqrt{48} = 8\sqrt{3}$

11. Show that $\sqrt[3]{40} - 3\sqrt[3]{320} + 4\sqrt[3]{135} = 2\sqrt[3]{5}$

12. Show that $\sqrt[3]{16} - 6\sqrt[3]{\tfrac{1}{4}} + 2\sqrt[3]{54} - 4\sqrt[3]{\tfrac{1}{32}} = 4\sqrt[3]{2}$.

13. $2\sqrt{\frac{a}{2}} - b\sqrt{\frac{a}{b}} = \sqrt{2a} - \sqrt{ab}$.

14. $5\sqrt[3]{a} - (4a)^{\frac{1}{2}} - 3a^{\frac{3}{2}} + \sqrt{16a} = 2a^{\frac{1}{3}}(a - 1)$.

15. $2\sqrt[3]{2x} + 6\sqrt[6]{4x^2} + \sqrt[9]{8x^3} = 9\sqrt[3]{2x}$

16. $\sqrt{18a^5b^3} + \sqrt{50a^3b^3} = (3a^2b + 5ab)\sqrt{2ab}$.

17. $\sqrt[3]{4} \times 7\sqrt[3]{6} \times \sqrt[3]{\frac{5}{8}} = 7\sqrt[3]{15}$.

18. $(\sqrt{5} + 1)(1 - \sqrt{5}) = -4$; and

$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1$.

19. $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \dfrac{a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b}{a - b}$; and

$\sqrt[6]{\dfrac{1}{x^3y^2}} \times \sqrt[5]{x^2} \cdot \sqrt[3]{y} = \dfrac{1}{x^{\frac{1}{10}}}$.

20. $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{2} + 1} = \sqrt{2} - \sqrt{3} + \sqrt{6} - 2$; and

$(\sqrt{3} + \sqrt{2})^3 = 9\sqrt{3} + 11\sqrt{2}$.

21. $\dfrac{1}{x - \sqrt{x^2 - 1}} + \dfrac{1}{x + \sqrt{x^2 - 1}} = 2x$; and

$\dfrac{\sqrt{x + y} + \sqrt{x - y}}{\sqrt{x + y} - \sqrt{x - y}} = \dfrac{x}{y} + \dfrac{\sqrt{x^2 - y^2}}{y}$.

22. Prove that $\dfrac{\sqrt{a^2 - x^2}}{2} + \dfrac{x}{2}$ is the square root of

$\dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{4}$; and that

$$\left\{\sqrt{\frac{x + \sqrt{x^2 - a}}{2}} + \sqrt{\frac{x - \sqrt{x^2 - a}}{2}}\right\}^2 = x + \sqrt{a}$$

CHAPTER V.

SIMPLE EQUATIONS.

1. $\sqrt{x + 16} = 2 + \sqrt{x}$; find x.

Squaring each member of the equation, we have

$$x + 16 = 4 + 4\sqrt{x} + x,$$

$$16 - 4 = 4\sqrt{x}, \qquad 12 = 4\sqrt{x},$$

$$3 = \sqrt{x}, \qquad \therefore\ 9 = x.$$

2. $\dfrac{x - a}{\sqrt{a} + \sqrt{x}} = \dfrac{\sqrt{x} - \sqrt{a}}{3} + 2\sqrt{a}$; find x.

First, reducing the left-hand member of the equation,

$$\frac{x - a}{\sqrt{a} + \sqrt{x}} = \frac{(\sqrt{x} + \sqrt{a})(\sqrt{x} - \sqrt{a})}{(\sqrt{x} + \sqrt{a})} = (\sqrt{x} - \sqrt{a});$$

$$\therefore\ (\sqrt{x} - \sqrt{a}) = \frac{\sqrt{x} - \sqrt{a}}{3} + 2\sqrt{a},$$

$$3(\sqrt{x} - \sqrt{a}) = (\sqrt{x} - \sqrt{a}) + 6\sqrt{a},$$

$$2(\sqrt{x} - \sqrt{a}) = 6\sqrt{a}, \qquad \sqrt{x} - \sqrt{a} = 3\sqrt{a},$$

$$\sqrt{x} = 4\sqrt{a}, \qquad x = 16a.$$

3. $\dfrac{x}{\sqrt{a^2 + x^2}} = \dfrac{c - x}{\sqrt{b^2 + (c - x)^2}}$; find x.

Squaring each member, we have

$$\frac{x^2}{a^2 + x^2} = \frac{(c - x)^2}{b^2 + (c - x)^2};$$

Multiply by the product of the denominators, which will thus both be cancelled.

$$b^2 x^2 + (c - x)^2 x^2 = a^2 (c - x)^2 + (c - x)^2 x^2,$$

$$b^2 x^2 = a^2 (c - x)^2, \qquad bx = a(c - x) = ac - ax,$$

$$ax + bx = ac, \qquad (a + b)x = ac,$$

$$\therefore\ x = \frac{ac}{a + b}.$$

4. Given $1 - \sqrt{1-x} = n(1 + \sqrt{1-x})$; find x.

$$1 - \sqrt{1-x} = n + n\sqrt{1-x},$$

$$1 - n = n\sqrt{1-x} + \sqrt{1-x}$$

$$= (n+1)\sqrt{1-x},$$

$$1 - 2n + n^2 = (n^2 + 2n + 1)(1-x)$$

$$= n^2 + 2n + 1 - (n^2 + 2n + 1)x,$$

$$(n+1)^2 x = 4n, \qquad \therefore\ x = \frac{4n}{(n+1)^2}.$$

5. $x - 4 = \dfrac{x^2}{(1 + \sqrt{1+x})^2}$; find x

$$\sqrt{x-4} = \frac{x}{\sqrt{1+x}+1}$$

$$= \frac{x(\sqrt{1+x}-1)}{(\sqrt{1+x}+1)(\sqrt{1+x}-1)} = \frac{x(\sqrt{1+x}-1)}{(1+x-1)}, \text{ Art. XIV.}$$

$$= \frac{x(\sqrt{1+x}-1)}{x} = \sqrt{1+x} - 1,$$

$$\therefore\ x - 4 = 1 + x - 2\sqrt{1+x} + 1,$$

$$2\sqrt{1+x} = 6, \qquad \sqrt{1+x} = 3,$$

$$1 + x = 9, \qquad \therefore\ x = 8.$$

6. $8\sqrt{3x} + \dfrac{243 + 324\sqrt{3x}}{16x - 3} = 16x + 3,$

$$16x - 8\sqrt{3}.\sqrt{x} + 3 = \frac{81(3 + 4\sqrt{3}.\sqrt{x})}{16x - 3}$$

$$= \frac{81\sqrt{3}.(\sqrt{3} + 4\sqrt{x})}{(4\sqrt{x} + \sqrt{3})(4\sqrt{x} - \sqrt{3})}$$

that is $(4\sqrt{x} - \sqrt{3})^2 = \dfrac{81\sqrt{3}}{(4\sqrt{x} - \sqrt{3})},$

$$(4\sqrt{x} - \sqrt{3})^3 = 3^4 \,.\, 3^{\frac{1}{2}} = 3^{\frac{9}{2}},$$

$$4\sqrt{x} - \sqrt{3} = 3^{\frac{3}{2}}, \quad 4\sqrt{x} = 3^{\frac{3}{2}} + 3^{\frac{1}{2}},$$

$$16x = 3^3 + 2\,.\,3^2 + 3 = 27 + 18 + 3 = 48, \ \therefore x = 3.$$

7. $\sqrt[3]{a + x} + \sqrt[3]{a - x} = \sqrt[3]{b}$; find x.

$(a+x)^{\frac{1}{3}} + (a - x)^{\frac{1}{3}} = b^{\frac{1}{3}}$; and, cubing each member (see p. 38),

$$a + x + a - x + 3\,\{(a + x)^{\frac{1}{3}} + (a - x)^{\frac{1}{3}}\}\,(a + x)^{\frac{1}{3}}\,(a - x)^{\frac{1}{3}} = b,$$

$$3\,\{(a + x)^{\frac{1}{3}} + (a - x)^{\frac{1}{3}}\}\,(a + x)^{\frac{1}{3}}\,(a - x)^{\frac{1}{3}} = b - 2a,$$

$$3\,b^{\frac{1}{3}}(a + x)^{\frac{1}{3}}(a - x)^{\frac{1}{3}} = b - 2a,$$

$$\because (a + x)^{\frac{1}{3}} + (a - x)^{\frac{1}{3}} = b^{\frac{1}{3}},$$

$$\sqrt[3]{(a + x)\,(a - x)} = \frac{b - 2a}{3\,b^{\frac{1}{3}}},$$

$$(a + x)\,(a - x) = \frac{(b - 2a)^3}{27b}, \quad a^2 - x^2 = \frac{(b - 2a)^3}{27b},$$

$$x^2 = a^2 - \frac{(b - 2a)^3}{27b}, \quad \therefore x = \sqrt{\left\{a^2 - \frac{(b - 2a)^3}{27b}\right\}}$$

8. $x\sqrt{(x^4 - 1)} + \sqrt[4]{(x^4 - 1)} = x^3.$

$$\sqrt[4]{(x^4 - 1)} = x^3 - x\sqrt{(x^4 - 1)},$$

$$\sqrt{(x^4 - 1)} = x^6 - 2x^4\sqrt{(x^4 - 1)} + x^6 - x^2,$$

$$x^2 + \sqrt{(x^4 - 1)} = 2x^6 - 2x^4\sqrt{(x^4 - 1)}$$

$$= 2x^4\,\{x^2 - \sqrt{(x^4 - 1)}\}.$$

Multiply by $\{x^2 + \sqrt{(x^4 - 1)}\}$;

$$\{x^2 + \sqrt{(x^4 - 1)}\}^2 = 2x^4\,(x^4 - x^4 + 1) = 2x^4$$

$$x^2 + \sqrt{(x^4 - 1)} = x^2\sqrt{2}, \quad \sqrt{(x^4 - 1)} = x^2(\sqrt{2} - 1).$$

$$x^4 - 1 = x^4\,(3 - 2\sqrt{2}) = 3x^4 - 2\sqrt{2}\,.\,x^4,$$

$$(2\sqrt{2} - 2)\,x^4 = 1, \quad x^4 = \frac{1}{2(\sqrt{2} - 1)}.$$

Multiply both terms of the fraction by $\sqrt{2}+1$.

$$x^4=\frac{\sqrt{2}+1}{2}; \quad \therefore x=\sqrt[4]{\frac{\sqrt{2}+1}{2}}.$$

9. $$\frac{a+x}{\sqrt{a}+\sqrt{a+x}}+\frac{a-x}{\sqrt{a}+\sqrt{a-x}}=\sqrt{a}; \text{ find } x.$$

$$\frac{(a+x)(\sqrt{a}-\sqrt{a+x})}{a-(a+x)}+\frac{(a-x)(\sqrt{a}-\sqrt{a-x})}{a-(a-x)}=\sqrt{a},$$

$$\frac{(a+x)(\sqrt{a}-\sqrt{a+x})}{-x}+\frac{(a-x)(\sqrt{a}-\sqrt{a-x})}{x}=\sqrt{a},$$

$$(a-x)(\sqrt{a}-\sqrt{a-x})-(a+x)(\sqrt{a}-\sqrt{a+x})=\sqrt{a}\,.\,x,$$

$$a\sqrt{a}-\sqrt{a}\,.\,x-(a-x)^{\frac{3}{2}}-a\sqrt{a}-\sqrt{a}\,.\,x+(a+x)^{\frac{3}{2}}=\sqrt{a}\,.\,x,$$

$$(a+x)^{\frac{3}{2}}-(a-x)^{\frac{3}{2}}=3\sqrt{a}\,.\,x,$$

$$(a+x)^3+(a-x)^3-2(a^2-x^2)^{\frac{3}{2}}=9ax^2,$$

$$2a^3+6ax^2-9ax^2=2(a^2-x^2)^{\frac{3}{2}},$$

$$2a^3-3ax^2=2(a^2-x^2)^{\frac{3}{2}},$$

$$4a^6-12a^4x^2+9a^2x^4=4(a^2-x^2)^3,$$

$$=4a^6-12a^4x^2+12a^2x^4-4x^6,$$

$$4x^6=3a^2x^4, \qquad 4x^2=3a^2,$$

$$2x=a\sqrt{3}, \qquad \therefore x=\frac{a}{2}\sqrt{3}.$$

XV. In solving equations involving *two* unknown quantities, as x and y, we may adopt any one of the three following methods, namely:

1. Eliminate one of the unknown quantities, and from the resulting equation, determine the value of the other.
2. Find a value of one of the unknown quantities, as x, from each equation, and then equate these two values.
3. Find a value of one of the unknown quantities, as x, and substitute it in the other equation.

1. Given $\left.\begin{aligned}5x + 3y &= 74\\ 3x + 2y &= 46\end{aligned}\right\}$ Find the values of x and y by elimination.

Multiplying the first equation by 2, the cofficient of y in the second; and multiplying the second by 3, the coefficient of y in the first, we have

$$10x + 6y = 148,$$
$$9x + 6y = 138.$$

Now, since y has the same coefficient and the same sign in each equation, if we *subtract** the lower equation from the upper, y will disappear, and we shall have

$$x = 10; \qquad \therefore 3x = 30.$$

And, writing 30 instead of $3x$ in the second given equation,

$$30 + 2y = 46, \qquad 2y = 16, \qquad \therefore y = 8.$$

When three or more equations are given they may be taken in pairs, and the *same* unknown quantity eliminated from each pair.

2 Given $\left.\begin{aligned}x^m y^n &= a, \quad (1)\\ x^p y^q &= b, \quad (2)\end{aligned}\right\}$ Find x and y by the second method.

$$\text{From (1)} \quad x^m = \frac{a}{y^n}; \quad \therefore x = \frac{a^{\frac{1}{m}}}{y^{\frac{n}{m}}};$$

$$\ldots \quad (2) \quad x^p = \frac{b}{y^q}; \quad \therefore x = \frac{b^{\frac{1}{p}}}{y^{\frac{q}{p}}}.$$

Hence, having two distinct values of the same quantity x, these values are equal to each other, and we may equate them; thus,

$$\frac{a^{\frac{1}{m}}}{y^{\frac{n}{m}}} = \frac{b^{\frac{1}{p}}}{y^{\frac{q}{p}}}, \quad \therefore \frac{y^{\frac{q}{p}}}{y^{\frac{n}{m}}} = \frac{b^{\frac{1}{p}}}{a^{\frac{1}{m}}}, \qquad \text{or } y^{\frac{q}{p}-\frac{n}{m}} = \frac{b^{\frac{1}{p}}}{a^{\frac{1}{m}}},$$

$$\text{or, } y^{\frac{mq-np}{mp}} = \frac{b^{\frac{1}{p}}}{a^{\frac{1}{m}}}, \quad \therefore y = \left(\frac{b^{\frac{1}{p}}}{a^{\frac{1}{m}}}\right)^{\frac{mp}{mq-np}} = \left(\frac{b^m}{a^p}\right)^{\frac{1}{mq-np}}$$

* When the signs of the quantity to be eliminated are unlike, we must, of course, *add* the two equations.

And proceeding in a similar manner, to obtain two distinct values of y from the given equations, and then equating them, we shall have

$$x = \left(\frac{a^q}{b^n}\right)^{\frac{1}{mq-np}}.$$

3. Given $\frac{y}{x} + \frac{3x}{x+y} = \frac{x^2-y^2}{y}$, (1)
and $\frac{x}{y} = \frac{x+y}{x} + \frac{y}{x}$, ... (2)
Find x and y by the third method.

$$(2)\ \frac{x}{y} - \frac{y}{x} = \frac{x+y}{x}, \qquad \frac{x^2-y^2}{xy} = \frac{x+y}{x},$$

and dividing each member by $\frac{x+y}{x}$, we have

$$\frac{x-y}{y} = 1, \ \therefore x - y = y,$$

$$\therefore \left.\begin{array}{l} x = 2y \\ x^2 = 4y^2 \end{array}\right\}$$ substituting these values in (1) we have

$$\frac{y}{2y} + \frac{6y}{2y+y} = \frac{4y^2-y^2}{y},$$

$$\frac{1}{2} + \frac{6}{2+1} = 4y - y, \qquad \frac{1}{2} + \frac{6}{3} = 3y,$$

$$\text{that is } \frac{5}{2} = 3y \quad \therefore 6y = 5,$$

$$\therefore y = \frac{5}{6}, \qquad \text{and } x = 2y = \frac{5}{3}.$$

4. Given $\frac{xy}{x+y} = 70$, $\frac{xz}{x+z} = 84$, $\frac{yz}{y+z} = 140$;
find x, y, and z.

From the first equation, by taking the reciprocal* of each

* The *reciprocal* of any quantity is *unity divided by that quantity;* thus, the reciprocals of x, $\frac{x}{c}$, and $\frac{a+b}{x}$ are $\frac{1}{x}$, $\frac{c}{x}$, and $\frac{x}{a+b}$ respectively.

member, we have $\frac{x+y}{xy}=\frac{1}{70}$, or $\frac{x}{xy}+\frac{y}{xy}=\frac{1}{70}$;

$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{70}$, (1); and similarly from the second and third.

$\frac{1}{x}+\frac{1}{z}=\frac{1}{84}$, (2)

$\frac{1}{y}+\frac{1}{z}=\frac{1}{140}$, (3)

(1) $\frac{1}{x}+\frac{1}{y}=\frac{2}{140}$, and subtracting (3) from this,

$\frac{1}{x}-\frac{1}{z}=\frac{1}{140}=\frac{3}{420}$,

(2) $\frac{1}{x}+\frac{1}{z}=\frac{5}{420}$,

$\therefore \frac{2}{x}=\frac{8}{420}=\frac{2}{105}$, $\therefore x=105$

and $\frac{2}{z}=\frac{2}{420}$; $\therefore z=420$.

(3) $\frac{1}{y}+\frac{1}{420}=\frac{1}{140}$; $\therefore \frac{1}{y}=\frac{3}{420}-\frac{1}{420}=\frac{2}{420}=\frac{1}{210}$:

$\therefore y=210$.

Solve the following Simple Equations.

1. $\sqrt{x^2+3}=\sqrt{7}$. Ans. $x=2$.
2. $\sqrt{x^2-16}=x-2$. Ans. $x=5$.
3. $\sqrt{x}-1=\sqrt{x-9}$. Ans. $x=25$.
4. $\sqrt{x}+\sqrt{x-3}=3$. Ans. $x=4$.
5. $\sqrt{x}-\sqrt{2}=\sqrt{x-2}$. Ans. $x=2$.
6 $\sqrt[3]{4x+3}=3$. Ans. $x=6$

7. $\sqrt{5x+4} = \sqrt{3x} + 2$ Ans. $x = 12$

8. $\sqrt{x^2+a^2} + x = b.$ Ans. $x = \frac{b^2 - a^2}{2b}.$

9 $2\sqrt{x} - \sqrt{a} = 2\sqrt{x-a}.$ Ans. $x = \frac{25a}{16}.$

10 $a + x = \sqrt{x^2 + 5x - a}.$ Ans. $x = \frac{a^2+a}{5-2a}.$

11. $\sqrt{a^2+x^2} = \sqrt[4]{b^4+x^4}.$ Ans. $x = \sqrt{\frac{b^4-a^4}{2a^2}}.$

12. $\sqrt{a-x} = \frac{a}{\sqrt{a-x}} - x.$ Ans. $x = a - 1.$

13. $\sqrt{a+x} + \sqrt{a-x} = 2\sqrt{x}.$ Ans. $x = \frac{4}{5}a.$

14. $\frac{\sqrt{x}-2}{3} + 3 = \frac{x-4}{\sqrt{x}+2}.$ Ans. $x = 42\frac{1}{4}.$

15. $x - \sqrt{(a^2 + \sqrt{x^2-1})} = a.$ Ans. $x = \frac{4a^2+1}{4a}$

16. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = c$ Ans. $x = \frac{2ac}{c^2+1}.$

17. $\sqrt{x+a} = \sqrt{a} + \sqrt{x-a}.$ Ans. $x = \frac{5a}{4}.$

18. $\sqrt{x} - \sqrt{a-x} = \frac{\sqrt{x}+\sqrt{a-x}}{2}.$ Ans. $x = \frac{9a}{10}.$

19. $\sqrt{\left(\frac{a^2}{x}+b\right)} - \sqrt{\left(\frac{a^2}{x}-b\right)} = c.$ Ans. $x = \frac{4a^2c^2}{4b^2+c^4},$

20. $\sqrt{x^2+9} = 2 + \frac{x^2-9}{\sqrt{x^2+9}+3}.$ Ans. $x = 3\sqrt{15}.$

21. $\frac{na^2}{\sqrt{x^2+a^2}} - x = \sqrt{a^2+x^2}.$ Ans. $x = \frac{(n-1)a}{(2n-1)^{\frac{1}{2}}}$

22. $ax - \sqrt{x^2 + x + 1} = \sqrt{x^2 - x + 1}$

Ans. $x = \frac{2}{a}\sqrt{\frac{a^2 - 1}{a^2 - 4}}$.

23. $\sqrt{a^2 + ax} = a - \sqrt{a^2 - ax}$. Ans. $x = \frac{a}{2}\sqrt{3}$.

24. $\frac{1}{x} + \frac{1}{a} = \sqrt{\left\{\frac{1}{a^2} + \sqrt{\left(\frac{4}{b^2 x^2} + \frac{1}{x^4}\right)}\right\}}$

Ans. $x = \frac{ab^2}{a^2 - b^2}$.

25. $\frac{ax - 1}{\sqrt{ax} + 1} - 4 = \frac{\sqrt{ax} - 1}{2}$. Ans. $x = \frac{81}{a}$.

26. $\frac{1}{2}\sqrt{x^2 + 3a^2} + \frac{1}{2}\sqrt{x^2 - 3a^2} = x\sqrt{a}$.

Ans. $x = \sqrt[4]{\frac{9a^3}{4 - 4a}}$.

27. $\frac{1}{\sqrt{1 - x} + 1} + \frac{1}{\sqrt{1 + x} - 1} = \frac{1}{x}$. Ans. $x = \frac{\sqrt{3}}{2}$.

28. $\sqrt[3]{1 + x} + \sqrt[3]{1 - x} = \sqrt[3]{2}$. Ans. $x = 1$.

29. $\frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} + \frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = 4x(x - 1)$.

Ans. $x = \frac{1}{2}$.

30. Given $\left.\begin{array}{r} x + y = 5 \\ \text{and } x - y = 1 \end{array}\right\}$ find x and y. Ans. $x = 3$. $y = 2$.

31. $\left.\begin{array}{r} 2x - y = 1. \\ x + 3y = 11. \end{array}\right\}$ Ans. $x = 2$. $y = 3$.

32. $\left.\begin{array}{r} 4x - 11y = 9. \\ 2x + 3y = 13. \end{array}\right\}$ Ans. $x = 5$. $y = 1$.

33. $\left.\begin{array}{r} 3x + 2y = 23. \\ 5y - 2x = 29. \end{array}\right\}$ Ans. $x = 3$. $y = 7$.

34. $\left.\begin{array}{l}\frac{x}{2} - y = 1. \\ x - \frac{y}{2} = 8.\end{array}\right\}$ Ans. $x = 10$. $y = 4$.

35. $\left.\begin{array}{l}\frac{x}{3} + \frac{y}{5} - 5 = 0 \\ 2x + \frac{y}{3} - 17 = 0.\end{array}\right\}$ Ans. $x = 6$. $y = 15$

36.

37. $\left.\begin{array}{l}\frac{x + y}{10} + \frac{x - y}{2} = 0. \\ \frac{x + y}{5} + \frac{x - y}{2} = 1.\end{array}\right\}$ Ans. $x = 4$. $y = 6$.

38. $\left.\begin{array}{l}\frac{2x - y}{4} - \frac{3}{2} = \frac{3y}{4} - x - 2. \\ \frac{x + y}{3} = 2\frac{2}{3}.\end{array}\right\}$ Ans. $x = 3$. $y = 5$.

39. $\left.\begin{array}{l}ax + by = c. \\ \frac{x}{b} - \frac{y}{a} = 1.\end{array}\right\}$ Ans. $x = \frac{ab + c}{2a}$. $y = -\frac{ab - c}{2b}$.

40. $\left.\begin{array}{l}\frac{x + 2}{y} = \frac{7}{8}. \\ \frac{x}{y - 2} = \frac{5}{6}.\end{array}\right\}$ Ans. $x = 5$. $y = 8$.

41. $\left.\begin{array}{l}\frac{m}{x} + \frac{n}{y} = a. \\ \frac{n}{x} + \frac{m}{y} = b.\end{array}\right\}$ Ans. $x = \frac{m^2 - n^2}{ma - nb}$. $y = \frac{m^2 - n^2}{mb - na}$.

42. $\left.\begin{array}{l}(x + 1)(y - 9) = (y + 7)(x + 5) - 112. \\ 3y - 2x = 9.\end{array}\right\}$

Ans. $x = 3$.
$y = 5$.

43. $$\left.\begin{aligned} y + \frac{x}{4} &= 10 - \frac{y - 2x - 1}{8}. \\ \frac{2x-1}{10} - \frac{6x - 2y}{5} &= \frac{x - y}{10}. \end{aligned}\right\}$$ Ans. $x = 4$, $y = 9$

44. $$\left.\begin{aligned} 3 \cdot 4x - {\cdot}02y &= {\cdot}01. \\ 2x + {\cdot}4y &= 1{\cdot}2. \end{aligned}\right\}$$ Ans. $x = {\cdot}02$, $y = 2{\cdot}9$

45. $$\left.\begin{aligned} \frac{x + ay}{3} &= c. \\ ax - by &= c. \end{aligned}\right\}$$ Ans. $x = \dfrac{(a + 3b)c}{a^2 + b}$. $y = \dfrac{(3a - 1)c}{a^2 + b}$.

46. $$\left.\begin{aligned} ax + by &= c^2. \\ a(a + x) &= b(b + y). \end{aligned}\right\}$$ Ans $x = \dfrac{b^2 + c^2 - a^2}{2a}$. $y = \dfrac{a^2 + c^2 - b^2}{2b}$.

47. $$\left.\begin{aligned} 2x - 2y + 3z &= 16. \\ 3x + 5y - 2z &= 6. \\ 4x + 3y - 4z &= -1. \end{aligned}\right\}$$ Ans. $x = 3$. $y = 1$. $z = 4$.

48. $$\left.\begin{aligned} \frac{x}{3} + \frac{y}{4} + \frac{z}{5} &= 47. \\ \frac{x}{4} + \frac{y}{5} + \frac{z}{6} &= 38. \\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} &= 62. \end{aligned}\right\}$$ Ans. $x = 24$. $y = 60$. $z = 120$.

49. $$\left.\begin{aligned} \frac{x + y}{z} &= 5. \\ \frac{y - z}{x} &= 1. \\ \frac{x - z}{y} &= \frac{z}{6}. \end{aligned}\right\}$$ Ans. $x = 4$. $y = 6$. $z = 2$.

50. $$\left.\begin{aligned} xyz &= 40. \\ xyw &= 80. \\ yzw &= 200. \\ xzw &= 100. \end{aligned}\right\}$$ Ans. $x = 2$. $y = 4$. $z = 5$. $w = 10$

Problems.

1. If to twice a certain number we add 9, the square root of the sum will be 5; what is the number? Ans. 8

2. If to five times a certain number we add 4, the square root of the sum will be equal to 2 + the square root of thrice the number; find it. Ans. 12.

3. There is a number, from the square of which if 7 be subtracted, the square root of the remainder will be equal to 7 diminished by the number itself; what is the number? Ans. 4.

4. The sum of two numbers is 56, and their difference is 24; find them. Ans. 40 and 16.

5. The sum of two numbers is 16, and the sum of their reciprocals = twice the difference of their reciprocals; find the numbers. Ans. 4 and 12.

6. In a naval engagement, one-third of the fleet was taken, one-sixth sunk, and two ships were burnt; one-seventh of the remainder was lost in a storm after the action, and 24 ships are left; how many ships were in the fleet at first? Ans. 60.

7. The ages of two persons are in the ratio of 3 to 4, but 10 years ago the ratio of their ages was that of 2 to 3; what are their ages? Ans. 30 and 40.

8. A wine merchant mixed 20 gallons of spirits at 9*s.* a gallon with 36 gallons at 11*s.* a gallon, and he now wishes to add as many gallons at 14*s.* a gallon as will make the mixture worth 12*s.* a gallon; how many gallons of this last must he add? Ans. 48 gallons.

9. A grazier buys 12 sheep and 20 lambs of one person for 29*l.*; and 10 sheep and 30 lambs of another at the same price for 33*l.* 10*s.*; what was the price of each? Ans. Sheep, 25*s.*, lambs, 14*s.*

10. A and B made a joint stock of 833*l.*, which, after a successful speculation, produced a clear gain of 153*l.* Of this B had 45*l.* more than A. What did each person contribute to the stock? Ans. A 294*l.*, and B 539*l.*

11. Suppose that for every 10 sheep a farmer kept, he should plough an acre of ground, and be allowed one acre of pasture for every 4 sheep; how many sheep may that person keep who farms 700 acres? Ans. 2000.

12. If the numerator of a certain fraction be added to the

denominator, the result will be equal to 5 times the numerator; what is the fraction? Ans. $\frac{1}{4}$.

13. What number is that to which, if 1, 5, and 13 be severally added, the first sum shall be to the second as the second to the third? Ans. 3.

14. A sets out from C to go to D, at the same time that B sets out from D to go to C; A arrives at D a hours, and B at C b hours, after they met; how long did each take to perform the journey? Ans. A $a^{\frac{1}{2}}(a^{\frac{1}{2}} + b^{\frac{1}{2}})$.
B $b^{\frac{1}{2}}(a^{\frac{1}{2}} + b^{\frac{1}{2}})$.

15. A person has two sorts of wine, one worth 20 pence a quart, and the other 12 pence, from which he would mix a quart to be worth 14 pence; how much of each must he take? Ans. $\frac{1}{4}$ of 1st; $\frac{3}{4}$ of 2nd.

16. A number consisting of two digits is equal to 5 times the sum of its digits, and if the sum of its digits be added to the number, it will be inverted; what is the number?
Ans. 45.

17. A person at play won twice as much as he began with, and then lost 16 shillings. After this he lost four-fifths of what remained, and then won as much as he began with, and counting his money, found he had 80 shillings; what sum did he begin with? Ans. 52 shillings.

18. A can perform a piece of work in a days, and B in b days; in how many days can they finish it, if they both work at it together?
Ans. $\frac{ab}{a+b}$ days.

19. A trader maintained himself for 3 years at the expense of 50*l.* a year, and in each of those years augmented that part of his stock which was not so expended by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled; what was that stock? Ans. 740*l.*

20. A man rows a boat, with the tide, 5 miles in $\frac{3}{4}$ of an hour, and returns against a tide half as strong, in $1\frac{1}{2}$ hour; required the velocity of the strongest tide.
Ans. $2\frac{2}{9}$ miles per hour.

21. Divide the number 11520 into 3 such parts that 9 times the sum of the first and second shall be equal to 7 times the sum of the second and third, and if the first be

subtracted from the second, 8 times the remainder shall be equal to the sum of the first and third.

Ans. 2880, 3840, and 4800.

22. A person bought a certain number of sheep for 94*l.*; having lost 7 of them, he sold one-fourth of the remainder at prime cost for 20*l.*; how many sheep had he at first?

Ans. 47.

23. A farmer pays his rent in certain fixed numbers of quarters of wheat and barley; when wheat is at 55*s.* and barley at 33*s.* a quarter, the portions of rent by wheat and barley are equal to one another; but when wheat is at 65*s.* and barley at 41*s.* a quarter, the rent is increased 7*l.*; what is the corn-rent? Ans. 6 qrs. wheat, 10 qrs. barley.

24. The circumference of the fore-wheel of a carriage is a feet, and that of the hind wheel b feet; what is the distance travelled by the carriage when the fore-wheel has made n revolutions more than the hind-wheel?

Ans. $\frac{abn}{b-a}$ feet.

CHAPTER VI.

QUADRATIC EQUATIONS.

XVI. A *quadratic* equation is one in which the unknown quantity is involved to the *second power*, or *square*. If the *square* alone is contained in the equation, it is called a *pure* quadratic; but if the *square* and *first power* are both contained in the equation, it is called an *adfected quadratic.* Thus, $x^2 = 4$, and $ay^2 - a = b$, are *pure* quadratics; and $x^2 + 2x = 8$, $y^2 - ay - c = 0$, are *adfected* quadratics.

Ex. 1. Solve the pure quadratic $7x^2 + 18 = 4x^2 + 450$.

By transposition, $7x^2 - 4x^2 = 450 - 18$,

$$\text{or } 3x^2 = 432,$$

$$\therefore x^2 = \frac{432}{3} = 144,$$

and extracting the square root of each member, we have

$$x = \pm 12.$$

The sign $\pm$ is prefixed to the value of x because

$+12 \times +12 = +144$, and $-12 \times -12 = +144$

Examples.

Solve the following pure quadratics

1. $x^2 = 144$. Ans. $x = \pm 12$.

2. $x^2 - 9 = 16$ Ans. $x = \pm 5$.

3. $\frac{3x^2}{4} - 5 = 7$. Ans. $x = \pm 4$.

4. $2\sqrt{1 - x^2} = \sqrt{3}$. Ans. $x = \pm \frac{1}{2}$.

5. $\frac{\sqrt{a^2 + x^2} - x}{x} = \frac{1}{b}$. Ans. $x = \frac{ab}{\sqrt{2b+1}}$.

6. $\frac{7x^2}{4} - \frac{4x^2+5}{2} + \frac{2x^2-15}{4} = 0$. Ans. $x = \pm 5$.

7. $\frac{1}{2x^2} + 7 = \frac{9}{4x^2}$. Ans. $x = \pm \frac{1}{2}$.

8. $35 - \frac{x^2+50}{5} = x^2 - \frac{x^2-10}{3}$. Ans. $x = \pm 5$

ADFECTED QUADRATICS.

XVII. Any adfected quadratic may be reduced to the form $x^2 + px = q$, where p and q represent known quantities or numbers; and if we add $\left(\frac{p}{2}\right)^2$ to each side of this equation, we shall have

$$x^2 + px + \left(\frac{p}{2}\right)^2 = \frac{p^2}{4} + q;$$

and taking the square root of each side, we have

$$x + \frac{p}{2} = \pm\sqrt{\left(\frac{p^2}{4} + q\right)};\ \therefore x = -\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} - q\right)}.$$

Hence this rule for the solution of an adfected quadratic "Add the *square of half the coefficient of the second term* to each side, and then extract the square root of each side."

Examples.

1. $x^2 + 8x = 20$. Complete the square.

$x^2 + 8x + 4^2 = 20 + 16 = 36$. Extract the root.

$x + 4 \quad = \pm 6;$

$\therefore\ x = \pm 6 - 4 = +6 - 4$, or $-6 - 4 = 2$, or -10.

2. $x^2 - 5x = 6$. Complete the square.

$x^2 - 5x + \left(\frac{5}{2}\right)^2 = 6 + \frac{25}{4} = \frac{24}{4} + \frac{25}{4} = \frac{49}{4}$. Extract the root.

$$x - \frac{5}{2} = \pm \frac{7}{2},$$

$$\therefore\ x = \frac{5 \pm 7}{2} = \frac{12}{2}, \text{ or } -\frac{2}{2} = 6, \text{ or } -1$$

It may be seen by these examples, that an easy way of taking the square root of the first side is to take the square roots of the first and third terms, and place between them the sign of the second term.

It also appears that a quadratic equation has two roots, or values of the unknown quantity.

Let the two roots of the equation $x^2 + px - q = 0$, be represented by α and β respectively, then

$$\alpha = -\frac{p}{2} + \sqrt{\left(\frac{p^2}{4} + q\right)},\ \beta = -\frac{p}{2} - \sqrt{\left(\frac{p^2}{4} + q\right)}.$$

$$\therefore\ \alpha + \beta = -p,\quad \alpha\beta = \frac{p^2}{4} - \left(\frac{p^2}{4} + q\right) = -q;$$

that is, the *sum of the roots* = the coefficient of the second term with its sign changed, and the *product of the roots* = the last term. These propositions are true of equations of any degree.

It is obvious that the two roots above found are *equal* only when $\sqrt{\left(\frac{p^2}{4} + q\right)} = 0$, for then $\alpha = -\frac{p}{2}$, and $\beta = -\frac{p}{2}$. When q is positive, both roots are *real;* but when q is negative, they are *real* and unequal only when $\frac{p^2}{4} > q$, for if $\frac{p^2}{4} < q$,

and q is negative, $\frac{p^2}{4} - q$ is negative, and $\sqrt{\left(\frac{p^2}{4} - q\right)}$ is *imaginary*, since a negative quantity has no square root.

Any equation of the form $ax^{2n} + bx^n = c$, where the index of the unknown quantity in the first term is *double* the index of the unknown quantity in the second term, may be solved as an adfected quadratic.

Equations of a higher degree may frequently be made to assume the *form* of pure or adfected quadratics, and may then be solved accordingly.

Examples

1. $x^4 - 7x^2 = 8$. Complete the square.

$x^4 - 7x^2 + \left(\frac{7}{2}\right)^2 = 8 + \frac{49}{4} = \frac{32}{4} + \frac{49}{4} = \frac{81}{4}$. Extract the root.

$$x^2 - \frac{7}{2} = \pm \frac{9}{2},\ x^2 = \frac{7 \pm 9}{2} = \frac{16}{2}, \text{ or } -\frac{2}{2} = 8, \text{ or } -1,$$

$$\therefore x = \pm \sqrt{8}, \text{ or } \pm \sqrt{-1} = \pm 2\sqrt{2}, \text{ or } \pm \sqrt{-1}.$$

This equation being a *biquadratic*, has four roots, namely, $+2\sqrt{2}$, $-2\sqrt{2}$, $+\sqrt{-1}$, and $-\sqrt{-1}$; the first two *real*, and the remaining two *imaginary*.

2. $\frac{1}{3}\sqrt{\left(x^2 + \frac{x}{2} + \frac{17}{2}\right)} + \frac{2x^2 + x}{4} = 15\frac{3}{4}$,

$\frac{1}{3}\sqrt{\left(x^2 + \frac{x}{2} + \frac{17}{2}\right)} + \frac{x^2}{2} + \frac{x}{4} = \frac{63}{4}$. Multiply by 2

$\frac{2}{3}\sqrt{\left(x^2 + \frac{x}{2} + \frac{17}{2}\right)} + x^2 + \frac{x}{2} = \frac{63}{2}$. Add $\frac{17}{2}$ to each side.

$$x^2 + \frac{x}{2} + \frac{17}{2} + \frac{2}{3}\sqrt{\left(x^2 + \frac{x}{2} + \frac{17}{2}\right)} = 40.$$

Now, $\because \left(x^2+\frac{x}{2}+\frac{17}{2}\right)$ is the square of

$\sqrt{\left(x^2+\frac{x}{2}+\frac{17}{2}\right)}$, the equation has assumed the adfected form, and we may complete the square;

$$\left(x^2+\frac{x}{2}+\frac{17}{2}\right)+\frac{2}{3}\sqrt{\left(x^2+\frac{x}{2}+\frac{17}{2}\right)}+\left(\frac{1}{3}\right)^2=\frac{361}{9},$$

$$\sqrt{\left(x^2+\frac{x}{2}+\frac{17}{2}\right)}+\frac{1}{3}=\pm\frac{19}{3},$$

$$\sqrt{\left(x^2+\frac{x}{2}+\frac{17}{2}\right)}=\frac{\pm 19-1}{3}=\frac{18}{3}=6, \text{ or } -\frac{20}{3}.$$

Squaring both sides of this equation, we have

$x^2+\frac{x}{2}+\frac{17}{2}=36$ or $\frac{400}{9}$, and, taking the value 36,

$x^2+\frac{1}{2}x=36-\frac{17}{2}=\frac{72-17}{2}=\frac{55}{2}$. Completing,

$$x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2=\frac{55}{2}+\frac{1}{16}=\frac{440}{16}+\frac{1}{16}=\frac{441}{16},$$

Extracting the root, $x+\frac{1}{4}=\pm\frac{21}{4}$,

$$\therefore x=\frac{\pm 21-1}{4}=\frac{20}{4} \text{ or } -\frac{22}{4}=5, \text{ or } -\frac{11}{2}.$$

Again, $x^2+\frac{1}{2}x=\frac{400}{9}-\frac{17}{2}=\frac{800-153}{18}=\frac{647}{18}$,

Completing, $x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2=\frac{647}{18}+\frac{1}{16}=\frac{5185}{144}$,

$$\therefore x+\frac{1}{4}=\pm\frac{\sqrt{5185}}{12}, \therefore x=-\frac{1}{4}\pm\frac{\sqrt{5185}}{12}.$$

Hence $x=5$ or $-\frac{11}{2}$ or $-\frac{3\pm\sqrt{5185}}{12}$

3. $x^3 - 3x = 2$. Multiply by x.

$x^4 - 3x^2 = 2x$, $x^4 - 2x^2 = x^2 + 2x$,

$x^4 - 2x^2 + 1 = x^2 + 2x + 1$,

$x^2 - 1 = x + 1$, $x^2 - x = 2$,

$x^2 - x + \left(\frac{1}{2}\right)^2 = 2 + \frac{1}{4} = \frac{9}{4}$, $x - \frac{1}{2} = \pm \frac{3}{2}$,

$\therefore x = \frac{1 \pm 3}{2} = \frac{4}{2}$; or $-\frac{2}{2}$, that is, 2; or -1.

4. $x^3 - 2x = 4$,

$x^4 - 2x^2 = 4x$. Add $4x^2$ to each side.

$x^4 + 2x^2 = 4x^2 + 4x$,

$x^4 + 2x^2 + 1 = 4x^2 + 4x + 1$. Extract the roots.

$x^2 + 1 = \pm (2x + 1)$, $x^2 = 2x$, $\therefore x = 2$.

$x^2 + 1 = -2x - 1$, $x^2 + 2x + 1 = -1$,

$x + 1 = \pm \sqrt{-1}$, $\therefore x = -1 \pm \sqrt{-1}$

5. $(x^2 - 5)^2 = (x - 3)^2 + (x + 1)^2$,

$x^4 - 10x^2 + 25 = x^2 - 6x + 9 + x^2 + 2x + 1$

$x^4 - 12x^2 + 4x + 15 = 0$,

$x^4 - 8x^2 = 4x^2 - 4x - 15$,

$x^4 - 8x^2 + 16 = 4x^2 - 4x + 1$. Extract the roots.

$x^2 - 4 = \pm (2x - 1)$, $x^2 - 2x = 3$,

$x^2 - 2x + 1 = 4$, $x - 1 = \pm 2$,

$\therefore x = 1 \pm 2 = 3$, or -1

Again, $x^2 - 4 = -2x + 1$, $x^2 + 2x = 5$,

$x^2 + 2x + 1 = 6$, $x + 1 = \pm \sqrt{6}$,

$\therefore x = \pm \sqrt{6} - 1$.

6. $x^3 - x^2 = 4$. Multiply by 4.

$4x^3 = 4x^2 + 16$. Add $x^4 + 4x^2$ to each side.

$x^4 + 4x^3 + 4x^2 = x^4 + 8x^2 + 16$. Extract the root.

$x^2 + 2x = x^2 + 4,$

$2x = 4, \therefore x = 2.$

7. $(x - 1)\sqrt{(2x - x^2)} = \frac{1}{2},$

$(x^2 - 2x + 1)(2x - x^2) = \frac{1}{4},$

$-x^4 + 4x^3 - 5x^2 + 2x = \frac{1}{4},$

$4x^4 - 16x^3 + 20x^2 - 8x = -1,$

$4x^4 - 16x^3 + 16x^2 + 4x^2 - 8x = -1,$

$(2x^2 - 4x)^2 + 2(2x^2 - 4x) + 1 = 0.$ Extract the root

$2x^2 - 4x + 1 = 0, \quad x^2 - 2x = -\frac{1}{2},$

$x^2 - 2x + 1 = 1 - \frac{1}{2} = \frac{1}{2}, \qquad x - 1 = \pm\frac{1}{\sqrt{2}},$

$\therefore x = 1 \pm \frac{1}{\sqrt{2}} = 1 \pm \frac{\sqrt{2}}{2}.$

8. $x - 2\sqrt{(x + 2)} = 1 + \sqrt[4]{(x^3 - 3x + 2)}$

$(x - 1) - 2\sqrt{(x + 2)} = \sqrt[4]{\{(x - 1)^2(x + 2)\}}$

$= \sqrt{(x - 1)}\sqrt[4]{(x + 2)}$

$(x - 1) - \sqrt[4]{(x + 2)}\sqrt{(x - 1)} = 2\sqrt{(x + 2)},$

$(x - 1) - \sqrt[4]{(x + 2)}\sqrt{(x - 1)} + \frac{1}{4}\sqrt{(x + 2)} =$

$2\sqrt{(x + 2)} + \frac{1}{4}\sqrt{(x + 2)} = \frac{9}{4}\sqrt{(x + 2)},$

$\sqrt{(x - 1)} - \frac{1}{2}\sqrt[4]{(x + 2)} = \pm\frac{3}{2}\sqrt[4]{(x + 2)},$

$\sqrt{(x - 1)} = 2\sqrt[4]{(x + 2)}, \qquad x - 1 = 4\sqrt{(x + 2)},$

$x^2 - 2x + 1 = 16x + 32, \quad x^2 - 18x = 31,$

$x^2 - 18x + 9^2 = 81 + 31 = 112 = 16 \times 7,$

$x - 9 = \pm 4\sqrt{7}, \quad \therefore x = 9 \pm 4\sqrt{7}.$

Again, $\sqrt{(x-1)} = -\sqrt[4]{(x+2)}$,

$$x^2 - 2x + 1 = x + 2, \qquad x^2 - 3x = 1,$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = 1 + \frac{9}{4} = \frac{13}{4}, \qquad x - \frac{3}{2} = \pm\frac{\sqrt{13}}{2},$$

$$\therefore x = \frac{3 \pm \sqrt{13}}{2} = \tfrac{1}{2}(3 \pm \sqrt{13}).$$

9. $\sqrt{(a^2 - x^2)} + x\sqrt{(a^2-1)} = a^2\sqrt{(1-x^2)}$,

$$a^2 - x^2 + x^2(a^2-1) + 2x\{(a^2-x^2)(a^2-1)\}^{\frac{1}{2}} = a^4(1-x^2),$$

$$a^2 - x^2 + a^2x^2 - x^2 + 2x\{a^4 - a^2x^2 - a^2 + x^2\}^{\frac{1}{2}} = a^4 - a^4x^2,$$

$$a^4 - a^2x^2 - a^2 + x^2 - 2x\sqrt{(a^4 - a^2x^2 - a^2 + x^2)} + x^2 = a^4x^2,$$

Extracting the square root, and transposing x,

$$\sqrt{(a^4 - a^2x^2 - a^2 + x^2)} = a^2x + x$$

$$a^4 - a^2x^2 - a^2 + x^2 = a^4x^2 + 2a^2x^2 + x^2,$$

$$a^4 - a^2 = a^4x^2 + 3a^2x^2, \qquad a^2 - 1 = a^2x^2 + 3x^2,$$

$$(a^2+3)x^2 = a^2 - 1, \qquad x^2 = \frac{a^2-1}{a^2+3}, \quad \therefore x = \sqrt{\frac{a^2-1}{a^2+3}}$$

10 $$\sqrt{\left(x - \frac{1}{x}\right)} + \sqrt{\left(1 - \frac{1}{x}\right)} = x,$$

$$\sqrt{\left(1 - \frac{1}{x}\right)} = x - \sqrt{\left(x - \frac{1}{x}\right)},$$

$$1 - \frac{1}{x} = x^2 - 2x\sqrt{\left(x - \frac{1}{x}\right)} + x - \frac{1}{x},$$

$$x^2 - 1 - 2x\sqrt{\frac{x^2-1}{x}} + x = 0,$$

$$x^2 - 1 - 2x^{\frac{1}{2}}\sqrt{(x^2-1)} + x = 0, \quad \text{Extract,}$$

$$\sqrt{(x^2-1)} - x^{\frac{1}{2}} = 0, \qquad \sqrt{(x^2-1)} = x^{\frac{1}{2}}, \; x^2 - 1 = x,$$

$$x^2 - x = 1, \qquad x^2 - x + \frac{1}{4} = 1 + \tfrac{1}{4} = \tfrac{5}{4},$$

$$x - \frac{1}{2} = \frac{\sqrt{5}}{2}, \qquad \therefore x = \frac{1 \pm \sqrt{5}}{2}.$$

11. $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} + \sqrt{\frac{1-a}{1+a}}\sqrt[4]{\frac{1-x}{1+x}} = 2\sqrt[4]{\frac{1-a^2}{(1+a)^2}}.$

$$\sqrt[4]{\frac{1+x}{1-x}} + \sqrt{\frac{1-a}{1+a}}\sqrt[4]{\frac{1-x}{1+x}} = 2\sqrt[4]{\frac{1-a}{1+a}} \times \sqrt[4]{\frac{1+x}{1-x}}$$

$$\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-a}{1+a}} = 2\sqrt[4]{\frac{1-a}{1+a}}\sqrt[4]{\frac{1+x}{1-x}},$$

$$\sqrt{\frac{1+x}{1-x}} - 2\sqrt[4]{\frac{1-a}{1+a}} \cdot \sqrt[4]{\frac{1+x}{1-x}} + \sqrt{\frac{1-a}{1+a}} = 0,$$

$$\sqrt[4]{\frac{1+x}{1-x}} - \sqrt[4]{\frac{1-a}{1+a}} = 0,$$

$$\sqrt[4]{\frac{1+x}{1-x}} = \sqrt[4]{\frac{1-a}{1+a}},$$

$$\frac{1+x}{1-x} = \frac{1-a}{1+a}, \quad \therefore x = -a.$$

12. $\frac{1+x^4}{(1+x)^4} = \frac{1}{2}.$

$$2 + 2x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4,$$

$x^4 - 4x^3 - 6x^2 - 4x + 1 = 0$ ...... (1). Divide by x^2.

$$x^2 - 4x - 6 - \frac{4}{x} + \frac{1}{x^2} = 0,$$

$x^2 + \frac{1}{x^2} - 4x - \frac{4}{x} = 6.$ Add 2 to each side.

$$x^2 + 2 + \frac{1}{x^2} - 4\left(x + \frac{1}{x}\right) = 8,$$

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x + \frac{1}{x}\right) + 2^2 = 8 + 4 = 12,$$

$$x + \frac{1}{x} - 2 = \pm 2\sqrt{3},$$

$$x^2 + 1 - 2x = \pm 2\sqrt{3} \,.\, x,$$

$$x^2 - (2 \pm 2\sqrt{3})x = -1,$$

$$x^2 - 2(1 \pm \sqrt{3})x + (1 \pm \sqrt{3})^2 = 1 \pm 2\sqrt{3} + 3 - 1$$

$$= 3 \pm 2\sqrt{3},$$

$$x - (1 \pm \sqrt{3}) = \pm \sqrt{(3 \pm 2\sqrt{3})},$$

$$\therefore x = 1 \pm \sqrt{3} \pm \sqrt{(3 \pm 2\sqrt{3})}.$$

13. $2x^3 - x^2 = 1$, Multiply by $2x + 1$,

$$4x^4 - x^2 = 2x + 1, \qquad 4x^4 = x^2 + 2x + 1,$$

$$2x^2 = x + 1, \qquad x^2 - \frac{x}{2} = \frac{1}{2},$$

$$x^2 - \frac{x}{2} + \frac{1}{16} = \frac{1}{16} + \frac{8}{16} = \frac{9}{16}, \qquad x - \frac{1}{4} = \frac{3}{4},$$

$$\therefore x = \tfrac{4}{4} = 1.$$

It must be noticed here that, by multiplying the proposed equation by $2x + 1$, a root, found by equating this extraneous factor to zero, has been introduced, viz. the root $x = -\frac{1}{2}$. This root, therefore, must not be included among the roots of the proposed equation. After the multiplication, the equation $4x^4 = x^2 + 2x + 1$ leads to the *two* equations $2x^2 = x + 1$, and $2x^2 = -x - 1$: the complete solution of the former is $x = 1$, $x = -\frac{1}{2}$; and that of the latter, $x = \frac{-1 \pm \sqrt{-7}}{4}$; therefore, omitting the extraneous root $x = -\frac{1}{2}$, the three roots of the proposed equation are 1, and $\frac{-1 \pm \sqrt{-7}}{4}$.

Solve the following Adfected Quadratics.

1. Given $x^2 - 8x = 9$, to find the values of x.

Ans. $x = 9$, or -1.

2. $x^2 + 12x - 16 = 92$. Ans. $x = 6$, or -18.

3. $x^2 - 3x = 10$. Ans. $x = 5$, or -2.

4. $x^2 - x + 3 = 45$. Ans. $x = 7$, or -6.

5. $5x^2 + x = 4$. Ans. $x = \frac{4}{5}$, or -1.

6. $2x^2 - x = 21$. Ans. $x = \frac{7}{2}$, or -3.

7. $5x^2 + 6x - 3 = 60$. Ans. $x = 3$, or $-\frac{21}{5}$.

8. $x - 1 = -\frac{1}{x}$. Ans. $x = \frac{1 \pm \sqrt{-3}}{2}$.

9. $(x - 12)(x + 2) = 0$. Ans. $x = 12$, or -2.

10. $3x^2 - 14x + 15 = 0$. Ans. $x = 3$, or $1\frac{2}{3}$.

11. $2x^2 - 11x = 21$. Ans. $x = 7$, or $-1\frac{1}{2}$.

12. $ax^2 - bx = c$. Ans. $x = \frac{b \pm \sqrt{(b^2 + 4ac)}}{2a}$.

13. $4x - \frac{14 - x}{x + 1} = 14$. Ans. $x = 4$, or $-\frac{7}{4}$.

14. $x^2 - 4ax = -7a^2$. Ans. $x = (2 \pm \sqrt{-3})\,a$.

15. $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9}$. Ans. $x = 3$, or $\frac{21}{11}$.

16. $x + \sqrt{(5x + 10)} = 8$. Ans. $x = 18$, or 3.

17. $\frac{3x + 4}{5} - \frac{30 - 2x}{x - 6} = \frac{7x - 14}{10}$. Ans. $x = 36$, or 12

18. $x + \sqrt{(10x + 6)} = 9$. Ans. $x = 25$, or 3

19. $(x + 2)^2 = 2x^2 + 8$.* Ans. $x = 2$.

20. $\frac{x + 22}{3} - \frac{9x - 6}{2} = \frac{4}{x}$. Ans. $x = 2$, or $\frac{12}{25}$.

21. $\frac{2x}{9} - 2 = \frac{3x - 16}{18} - \frac{4x - 3}{4x + 3}$. Ans. $x = 6$, or $-4\frac{3}{4}$.

22. $\frac{x - 3}{x + 5} - \frac{x + 4}{x - 7} = 2\frac{7}{9}$. Ans. $x = 4$, or $-8\frac{21}{25}$.

* This will be found to be reducible to a pure quadratic.

23. $x^2 - (a + b)x + ab = 0$. Ans. $x = a$, or b.

24. $4x + 4\sqrt{(x + 2)} = 7$. Ans. $x = 4\frac{1}{4}$, or $\frac{1}{4}$.

25. $x = \dfrac{x - 9}{x^{\frac{1}{2}} + 3} + 15$. Ans. $x = 9$, or 16

26. $\sqrt{(x + 6)} + \sqrt{(x + 3)} = 3\sqrt{x}$. $x = \dfrac{9 \pm \sqrt{76}}{5}$.

27. $\dfrac{\sqrt{(4x + 20)}}{4 + \sqrt{x}} = \dfrac{4 - \sqrt{x}}{\sqrt{x}}$. Ans. $x = 4$, or $-\dfrac{64}{3}$.

28. $\sqrt{x} + 2 = \sqrt{(7 + 2x)}$. Ans. $x = 9$, or 1.

29. $\dfrac{x}{x + 1} + \dfrac{x + 1}{x} = 2\frac{1}{6}$. Ans. $x = 2$, or -3.

30. $\dfrac{4x^2}{3} = \dfrac{x}{3} + 11$. Ans. $x = 3$, or $-2\frac{3}{4}$.

31. $\sqrt{(x - a)} + \sqrt{(x + b)} = 2\sqrt{x}$.* $x = \dfrac{(a + b)^2}{8(b - a)}$.

32. $\dfrac{\sqrt{(a^2 x + b)}}{a + \sqrt{x}} = \dfrac{a - \sqrt{x}}{\sqrt{x}}$.

Ans. $x = \dfrac{-(2a^2 + b) \pm \sqrt{(4a^6 + 4a^2 b + b^2)}}{2(a^2 - 1)}$.

33. $\dfrac{x + 4}{3} - \dfrac{4x + 7}{9} = \dfrac{7 - x}{x - 3} - 1$. Ans. $x = 21$, or 5.

34. $\sqrt{(5a + x)} + \sqrt{(5a - x)} = \dfrac{12a}{\sqrt{(5a + x)}}$.

Ans. $x = 4a$, or $3a$.

35. $x^4 - 8x^2 = 9$. Ans. $x = \pm 3$, or $\pm\sqrt{-1}$.

36. $x^6 - 4x^3 = 32$. Ans. $x = 2$, or $\sqrt[3]{-4}$.

37. $\dfrac{nx + b}{\sqrt{x}} = \dfrac{na + b}{\sqrt{a}}$. Ans. $x = a$, or $\dfrac{b^2}{an^2}$.

38. $x^4 - 2x^2 = 3$. Ans. $x = \pm\sqrt{3}$, or $\pm\sqrt{-1}$.

39 $\sqrt{(4a + x)} + \sqrt{(a + x)} = 2\sqrt{(2a + x)}$.* $x = -\frac{7}{8}a$.

* This is reducible to a simple equation.

40. $x^3 - x^{\frac{3}{2}} = 56$. Ans. $x = 4$, or $\sqrt[3]{49}$.

41. $x + 5 = \sqrt{(x+5)} + 6$. Ans. $x = 4$, or -1.

42. $\sqrt{(2x+1)} + 2\sqrt{x} = \dfrac{21}{\sqrt{(2x+1)}}$.

Ans. $x = 4$, or -25.

43. $x^6 + 20x^3 = 69$. Ans. $x = \sqrt[3]{3}$, or $\sqrt[3]{-23}$.

44. $\dfrac{a+x}{\sqrt{(a-x)}} + \dfrac{a-x}{\sqrt{(a+x)}} = 2\sqrt{a}$.

Ans. $x = \pm a\sqrt{(8\sqrt{2} - 11)}$.

45. $x\sqrt{\left(\dfrac{a}{x} - 1\right)} = \sqrt{(x^2 - b^2)}$.

Ans. $x = \frac{1}{4}(a \pm \sqrt{a^2 + 8b^2})$.

46. $(a+1)(x-1)^2 = 2(x^2+1)$.

Ans. $x = \dfrac{\sqrt{a}+1}{\sqrt{a}-1}$, or $\dfrac{\sqrt{a}-1}{\sqrt{a}+1}$.

47. $x^2 - 7x + \sqrt{(x^2 - 7x + 18)} = 24$.

Ans. $x = 9$, or -2, or $\dfrac{7 \pm \sqrt{173}}{2}$.

48. $\left(\dfrac{1}{x}\right)^3 + a = \left(\dfrac{1}{x}\right)^{\frac{3}{2}}$. Ans. $x = \left\{\dfrac{1 \mp \sqrt{(1-4a)}}{2}\right\}^{\frac{2}{3}}$.

49. $5^x + \dfrac{125}{5^x} = 30$. Ans. $x = 2$, or 1.

50. $\sqrt{(a+x)} + \sqrt{(a-x)} = \dfrac{b}{\sqrt{(a+x)}}$.

Ans. $x = \frac{1}{2}\{b - a \pm \sqrt{(a^2 + 2ab - b^2)}\}$.

51. $x^2 - 2x + 6\sqrt{(x^2 - 2x + 5)} = 11$.

Ans. $x = 1$, or $1 \pm 2\sqrt{15}$.

52. $x^2(x-1) = 8(x+2)$. Ans. $x = \dfrac{\sqrt{-7} - 3}{2}$.

53. $x^4 - 2x^3 + x = a$. Ans. $x = \dfrac{1 \pm \sqrt{\{3 \pm 2\sqrt{(4a+1)}\}}}{2}$

54. $\dfrac{x}{\sqrt{x}+\sqrt{(a-x)}}+\dfrac{x}{\sqrt{x}-\sqrt{(a-x)}}=\dfrac{b}{\sqrt{x}}.$

Ans. $x=\dfrac{b\pm\sqrt{(b^2-2ab)}}{2}.$

55. $x^{-3}+\dfrac{1}{x\sqrt{x}}=2.$ Ans. $x=1$, or $\dfrac{\sqrt[3]{2}}{2}.$

56. $ax-b^2=x^2-2b\sqrt{(a^2-ax+x^2)}.$

Ans. $x=\dfrac{a\pm\sqrt{(a^2\pm8ab+4b^2)}}{2}$

57. $9x-3x^2+4\sqrt{(x^2-3x+5)}=11.$

Ans. $x=\dfrac{3\pm\sqrt{5}}{2}$

58. $\sqrt{\dfrac{a}{x}}+\sqrt{\dfrac{x}{a}}=\dfrac{\sqrt{(2ax-x^2)}}{x}.$

Ans. $x=-\frac{1}{2}a(3\mp\sqrt{13}).$

59. $x^2+\dfrac{1}{x^2}+x+\dfrac{1}{x}=4.$ Ans. $x=1$, or $\dfrac{-3\pm\sqrt{5}}{2}.$

60. $x=\dfrac{12+8x^{\frac{1}{2}}}{x-5}.$ Ans. $x=\dfrac{-3\mp\sqrt{(-7)}}{2}.$

61. $\sqrt{(x^2+1)}-\sqrt{(x^2-1)}=\dfrac{x}{\sqrt{(x^4-1)}}.$

Ans. $x=\pm\sqrt{\dfrac{2}{\sqrt{3}}}$

62. $x^3-6x-4=5.$ Ans. $x=3$, or $\dfrac{-3\pm\sqrt{-3}}{2}$

63. $3x^2=\dfrac{2}{x}+\dfrac{13}{3}.$ Ans. $x=-\dfrac{2}{3}$, or $\dfrac{1\pm\sqrt{10}}{3}$

64. $ax+2\sqrt{(n^2x+nax^2)}=(3x-1)n.$

Ans. $x=\dfrac{n}{n-a}$, or $\dfrac{n}{9n-a}$

65. $2(\sqrt{1-x}+1)=\dfrac{x}{\sqrt{1+x}-1}$. Ans. $x=\dfrac{24}{25}$.

66. $(x-b)^{2n}+2b^n x=b^{n+1}(2+b^{n-1})+2(x-b)^{n+1}$

Ans $x=2b$

67. $\left(x-\dfrac{ab}{x}\right)^2=\dfrac{a^2+ab}{2}\cdot\left(\dfrac{a^2}{x^2}+1\right)$.

Ans. $x=\sqrt{(a^2+2ab)}$, or $\sqrt{\dfrac{ab-a^2}{2}}$.

68. $x^4=-1$. $x=\sqrt[4]{-1}$, or $\dfrac{\pm1\pm\sqrt{-1}}{\sqrt{2}}$.

69. $x^{4n}-2x^{3n}+x^n=6$.

Ans. $x=\sqrt[n]{(\frac{1}{2}\pm\frac{1}{2}\sqrt{13})}$, or $\sqrt[n]{(\frac{1}{2}\pm\frac{1}{2}\sqrt{-7})}$

70. $\{(x-2)^2-x\}^2-90+x=(x-2)^2$.

Ans. $x=6$, -1, or $\dfrac{5\pm3\sqrt{-3}}{2}$.

71. $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2}=\sqrt[m]{(1-x^2)}$.

Ans. $x=\dfrac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}$

72. $8x^4+4x^3-18x^2+11x-2=0$.

Ans. $x=\frac{1}{2}$, or -2.

73. $4x^2-20-5\left(x+\dfrac{3}{x}\right)=-\dfrac{36}{x^2}$.

Ans. $x=3$, 1, or $\dfrac{-11\pm\sqrt{-71}}{8}$.

74. $\dfrac{2}{(x-4)^{\frac{3}{2}}}+\dfrac{(x-4)^{\frac{1}{2}}}{2}=\dfrac{17}{4(x-4)^{\frac{1}{2}}}$.

Ans. $x=12$, or $4\frac{1}{2}$

75 $x^4 + 4x^3 + 3x^2 + 4x + 1 = 0.$

$$\text{Ans. } x = \frac{-2 \pm \sqrt{3} \pm \sqrt{(3 \mp 4\sqrt{3})}}{2}$$

76. $(a^{2m} + 1)(x^{\frac{1}{2}} - 1)^2 = 2(x + 1).$

$$\text{Ans. } x = \left(\frac{a^{2m} \pm 1}{a^{2m} \mp 1}\right)^2.$$

77. $(x^2 - 9)^2 - 3 = 11(x^2 - 2).$ Ans. $x = \pm 5$, or ± 2.

78. $\sqrt{\{x + \sqrt{(2x - 1)}\}} - \sqrt{\{x - \sqrt{(2x - 1)}\}} = \frac{3}{5}\sqrt{\frac{10x}{x + \sqrt{(2x - 1)}}}.$ Ans. $x = \frac{5}{2}$, or $\frac{5}{8}$.

79. $\frac{a - \sqrt{(2ax - x^2)}}{a + \sqrt{(2ax - x^2)}} = \frac{x}{a - x}.$ Ans. $x = a$, or $\frac{a}{5}$.

80. $\frac{x^2 - 18}{x - 12} = \frac{4}{x}.$ Ans. $x = 4$, -2, or $-1 \pm \sqrt{7}$.

81. $x + 4 + \left(\frac{x + 4}{x - 4}\right)^{\frac{1}{2}} = \frac{12}{x - 4}.$

Ans. $x = \pm 5$, or $4\sqrt{2}$.

82. $(x + 119)^{\frac{1}{3}} + (70 - x)^{\frac{1}{3}} = 9.$ Ans. $x = 6$.

83. $x^4 + x^3 - 4x^2 + x + 1 = 0.$

$$\text{Ans. } x = 1, \text{ or } \frac{-3 \pm \sqrt{5}}{2}$$

84. $2x^2 + \sqrt{(x^2 + 9)} = x^4 - 9.$

$$\text{Ans. } x = \sqrt{\frac{3 \pm \sqrt{41}}{2}}, \text{ or } \sqrt{\frac{1 \pm \sqrt{37}}{2}}$$

85. $(a + x)\sqrt{(a^2 + x^2)} = 6(a - x)^2.$

$$\text{Ans. } x = \frac{9 \pm 4\sqrt{2}}{7}.a.$$

86. $(x + 3)^2 - 2(x^2 + 3) = 2x(x + 1)^2.$

Ans. $x = 1$, -3, or $-\frac{1}{2}$

87. $\dfrac{x}{x^2 + 4x} + \dfrac{x}{x^2 - 3x} = 1\frac{1}{8}$. Ans. $x = 4$, or $-1\frac{2}{3}$.

88. $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0$

Ans. $x = 2,\ 4,\ -3$, or $\pm\sqrt{-1}$

89. $\dfrac{x^2 + 1}{x} - \dfrac{1}{\sqrt{5}} \cdot \dfrac{x - 1}{\sqrt{x}} = 4\frac{2}{5}$.

Ans. $x = 5, \dfrac{1}{5}$, or $\dfrac{16 \mp 3\sqrt{29}}{10}$

90. $16(x^2 + 2)^{\frac{3}{2}} + \dfrac{3}{\sqrt{(x^2 + 2)}} = 32x^2 + 48$. $x = \pm\frac{1}{2}$

91. $x^{2m} - a = \dfrac{a}{x^m} + 1$. Ans. $x = \sqrt[m]{\{\frac{1}{2} \pm \sqrt{(a + \frac{1}{4})}\}}$.

92. $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0$.

Ans. $x = 5,\ -1$, or $2 \pm \sqrt{5}$.

93. $\dfrac{1 + x^3}{(1 + x)^3} = \frac{1}{3}$ Ans. $x = 2$, or $\frac{1}{2}$.

94 $\dfrac{(x - a)^2}{\sqrt{x}} + 2(x - a) = \dfrac{a^2}{\sqrt{x}} + 1$

Ans. $x = 2a + \frac{3}{2} \pm \sqrt{(2a + \frac{5}{4})}$.

95 $\sqrt{(x^2 - 1)} + \dfrac{x\sqrt{(x - 1)}}{\sqrt{(x + 1)}} = \dfrac{\sqrt{(x + 1)^3}}{\sqrt{(x - 1)}}$.

Ans. $x = \dfrac{3 \pm \sqrt{7}}{2}$

96. $(1 + x^3)(1 + x^2)(1 + x) = 30x^3$.

Ans. $x = \dfrac{3 \pm \sqrt{5}}{2}$.

97. $\dfrac{x^4}{2} + \dfrac{17x^3}{4} - 17x = 8$. Ans. $x = -8$, or $-\frac{1}{2}$

98. $x^2(x^2 - 23) = 10x(x^2 - 24) + 649$.

Ans. $x = \frac{1}{2}(5 \pm 3\sqrt{29})$

99. $\dfrac{1+x^3}{(1+x)^3} + \dfrac{1-x^3}{(1-x)^3} = a.$

Ans. $x = \left\{ \dfrac{a + 4 \mp 2\sqrt{3(a+1)}}{a-2} \right\}^{\frac{1}{2}}.$

100. $x + 7\sqrt[3]{x} = 22.$ $\quad x = 8$, or $\{-1 \pm \sqrt{-10}\}^3.$

101. $\sqrt{x} - \dfrac{7}{\sqrt{x} - 2} = \dfrac{8}{x}.$ Ans. $x = 1$, or 16.

102. $x^4\sqrt{x} + 2x^3 + 35x\sqrt{x} + 34 = \dfrac{72}{x\sqrt{x}}.$ Ans. $x = 1.$

QUADRATICS WITH TWO OR MORE UNKNOWN QUANTITIES.

Examples.

1. $\left.\begin{aligned} x + y &= 10 \\ xy &= 16 \end{aligned}\right\}\begin{aligned} &\ldots\ldots(1). \\ &\ldots\ldots(2). \end{aligned}$ Square (1), and multiply (2) by 4.

$$\left.\begin{aligned} x^2 + 2xy + y^2 &= 100 \\ 4xy &= 64 \end{aligned}\right\} \quad \text{Subtract.}$$

$$x^2 - 2xy + y^2 = 36 \quad \text{Extract the square root.}$$

$$(1) \quad \left.\begin{aligned} x - y &= \pm 6 \\ x + y &= 10 \end{aligned}\right\} \quad \text{Add and subtract.}$$

$$\begin{aligned} 2x &= 16, \text{ or } 4; \quad \therefore x = 8, \text{ or } 2. \\ 2y &= 4, \text{ or } 16; \quad \therefore y = 2, \text{ or } 8. \end{aligned}$$

This manner of proceeding may be adopted whenever the *sum* and *product* of the two unknown quantities are given

2. $\left.\begin{aligned} x - y &= 3 \\ xy &= 10 \end{aligned}\right\}\begin{aligned} &\ldots\ldots(1). \\ &\ldots\ldots(2). \end{aligned}$ Square (1), and multiply (2) by 4.

$$\left.\begin{aligned} x^2 - 2xy + y^2 &= 9 \\ 4xy &= 40 \end{aligned}\right\} \quad \text{Add.}$$

$$x^2 + 2xy + y^2 = 49 \quad \text{Extract the square root}$$

$$(1) \quad \left.\begin{aligned} x + y &= \pm 7 \\ x - y &= 3 \end{aligned}\right\} \quad \text{Add and subtract.}$$

$$\begin{aligned} 2x &= 10, \text{ or } -4; \quad \therefore x = 5, \text{ or } -2. \\ 2y &= 4, \text{ or } -10; \quad \therefore y = 2, \text{ or } -5 \end{aligned}$$

This mode of proceeding may be adopted whenever the *difference* and *product* of the two unknown quantities are given.

3. $\left.\begin{aligned} x^2 + y^2 &= 202 \\ x + y &= 20 \end{aligned}\right\}$ ......(1). ......(2). Double (1) and square (2).

$$\left.\begin{aligned} 2x^2 \quad + \quad 2y^2 &= 404 \\ x^2 + 2xy + y^2 &= 400 \end{aligned}\right\} \text{ Subtract.}$$

$$x^2 - 2xy + y^2 = 4$$

$$\left.\begin{aligned} \therefore x - y &= \pm 2 \\ x + y &= 20 \end{aligned}\right\} \quad \begin{aligned} \therefore x &= 11, \text{ or } 9. \\ y &= 9, \text{ or } 11. \end{aligned}$$

4. $\left.\begin{aligned} x^2 + y^2 &= 394 \\ x - y &= 2 \end{aligned}\right\}$ ......(1). ......(2). Double (1) and square (2).

$$\left.\begin{aligned} 2x^2 \quad + \quad 2y^2 &= 788 \\ x^2 - 2xy + y^2 &= 4 \end{aligned}\right\} \text{ Subtract.}$$

$$x^2 + 2xy + y^2 = 784$$

$$\left.\begin{aligned} \therefore x + y &= \pm 28 \\ x - y &= 2 \end{aligned}\right\} \quad \begin{aligned} \therefore x &= 15, \text{ or } -13. \\ y &= 13, \text{ or } -15. \end{aligned}$$

This mode of proceeding may always be adopted when the *sum of the squares* and the *sum* or *difference* of the two unknown quantities are given.

5. $\left.\begin{aligned} x^3 + y^3 &= 407 \\ x + y &= 11 \end{aligned}\right\}$ ......(1). ......(2). Divide (1) by (2).

$x^2 - xy + y^2 = 37$ Square (2).

$x^2 + 2xy + y^2 = 121$ Subtract the upper from the lower.

$3xy = 84 \quad \therefore xy = 28.$

$$\left.\begin{aligned} x^2 - xy + y^2 &= 37 \\ xy &= 28 \end{aligned}\right\} \text{ Subtract.}$$

$$x^2 - 2xy + y^2 = 9$$

$$\left.\begin{aligned} x - y &= \pm 3 \\ x + y &= 11 \end{aligned}\right\} \quad \begin{aligned} \therefore x &= 7, \text{ or } 4. \\ y &= 4, \text{ or } 7. \end{aligned}$$

If $x^5 + y^5 = a$, $x + y = b$, or $x^n - y^n = a$, $x - y = b$, n being an odd number, or any equations similar to these were given, we might proceed by dividing one equation by the other, as in this example.

6. $\left.\begin{array}{l} x^4 + y^4 = 337 \\ x + y = \quad 7 \end{array}\right\} \begin{array}{l} \ldots\ldots(1). \\ \ldots\ldots(2). \end{array}$ Raise (2) to the 4th power.

$$\left.\begin{array}{ll} & x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 2401 \\ (1) & x^4 \qquad\qquad\qquad\qquad\qquad\quad + y^4 = 337 \end{array}\right\} \text{ Subtract.}$$

$$4x^3y + 6x^2y^2 + 4xy^3 = 2064 \text{ Divide by } 4xy$$

$$x^2 + \tfrac{3}{2}xy + y^2 = \frac{516}{xy} \quad \text{Square (2).}$$

$$x^2 + 2xy + y^2 = 49 \quad \text{Subtract.}$$

$$\therefore \quad \tfrac{1}{2}xy = 49 - \frac{516}{xy}.$$

Multiply by $2xy$, and transpose.

$$x^2y^2 - 98xy = -1032. \quad \text{Complete the square.}$$

$$x^2y^2 - 98xy + 49^2 = 2401 - 1032 = 1369,$$

$$xy - 49 = \pm 37;$$

$$\therefore \quad xy = 49 \pm 37 = 86, \text{ or } 12.$$

Taking $xy = 12$, and $x + y = 7$, and proceeding as in *Ex*. 1, we have $x = 4$, or 3; $y = 3$, or 4.

7. $\left.\begin{array}{r} x^2 + y^2 + x + y = 922 \\ \sqrt{xy} = 20 \end{array}\right\} \begin{array}{l} \ldots\ldots(1). \\ \ldots\ldots(2). \end{array}$

$$xy = 400$$

$$2xy = 800. \quad \text{Adding this to (1),}$$

$$x^2 + 2xy + y^2 + x + y = 1722. \quad \text{Complete the square.}$$

$$(x + y)^2 + (x + y) + \frac{1}{4} = 1722 + \frac{1}{4} = \frac{6889}{4},$$

$$x + y + \frac{1}{2} = \pm \frac{83}{2};$$

$$\therefore x + y = \frac{\pm 83 - 1}{2} = \frac{82}{2}, \text{ or } -\frac{84}{2} = 41, \text{ or } -42;$$

and $\because xy = 400$, by proceeding as in *Ex*. 1, we have $x = 25$, $y = 16$.

8. $\left.\begin{array}{l} x^2 + 3y + x = 75 - 2xy \\ y^2 - y + x = 24 \end{array}\right\}\begin{array}{l}\ldots\ldots(1). \\ \ldots\ldots(2).\end{array}$

$$\left.\begin{array}{rl} x^2 + 2xy \quad + \quad x + 3y = & 75 \\ y^2 + \quad x - y \quad = & 24 \ldots\ldots(3), \end{array}\right\} \text{ Add}$$

$$x^2 + 2xy + y^2 + 2x + 2y = 99,$$

$$(x + y)^2 + 2(x + y) = 99, \text{ Complete the square.}$$

$$(x + y)^2 + 2(x + y) + 1 = 100,$$

$$x + y + 1 = \pm 10,$$

$$x + y = \pm 10 - 1 = 9, \text{ or } -11;$$

$$\therefore x = 9 - y. \quad \text{Substitute in (3).}$$

$$y^2 - y + 9 - y = 24, \qquad y^2 - 2y = 15,$$

$$y^2 - 2y + 1 = 16, \qquad y - 1 = \pm 4;$$

$$\therefore y = 1 \pm 4 = 5, \text{ or } -3; \text{ and } x = 4, \text{ or } 12.$$

9. $\left.\begin{array}{l} x^2 - y^2 = xy \\ x^2 + y^2 = x^3 - y^3 \end{array}\right\}\begin{array}{l}\ldots\ldots(1). \\ \ldots\ldots(2).\end{array}$ Multiply (1) by $x + y$

$$\left.\begin{array}{l} x^3 + x^2y - xy^2 - y^3 = xy(x + y) \\ x^3 \qquad\qquad\qquad - y^3 = x^2 + y^2 \end{array}\right\} \text{ Subtract.}$$

$$x^2y - xy^2 = x^2y + xy^2 - x^2 - y^2, \text{ Transpose}$$

$$\left.\begin{array}{l} x^2 + y^2 = 2xy^2 \\ x^2 - y^2 = xy \end{array}\right\} \text{ Add.}$$

$$2x^2 = 2xy^2 + xy. \quad \text{Divide by } 2x.$$

$$\left.\begin{array}{l} x = y^2 + \dfrac{y}{2} \\ x^2 = y^4 + y^3 + \dfrac{y^2}{4} \end{array}\right\} \begin{array}{l}\text{Substitute these} \\ \text{values of } x \\ \text{and } x^2 \text{ in (1).}\end{array}$$

$$y^4 + y^3 + \frac{y^2}{4} - y^2 = y^3 + \frac{y^2}{2},$$

$$y^4 = y^2 + \frac{y^2}{2} - \frac{y^2}{4},$$

$$y^4 = \frac{5y^2}{4}, \quad y^2 = \frac{5}{4}, \quad \therefore y = \frac{\sqrt{5}}{2},$$

$$\text{and } x = \frac{5}{4} + \frac{\sqrt{5}}{4} = \frac{\sqrt{5}(\sqrt{5} + 1)}{4}.$$

10. $\left.\begin{aligned} 5 + \sqrt{xy} &= 4\sqrt{y+2} \\ 8 + x &= \sqrt{128 + 64y} \end{aligned}\right\}\begin{aligned} &\ldots\ldots (1). \\ &\ldots\ldots (2). \end{aligned}$

(2) $8 + x = \sqrt{64(2+y)}$,

$$\left.\begin{aligned} 8 + x &= 8\sqrt{y+2} \\ (1)\quad 10 + 2x^{\frac{1}{2}}y^{\frac{1}{2}} &= 8\sqrt{y+2} \end{aligned}\right\} \text{ Subtract.}$$

$$2 + 2x^{\frac{1}{2}}y^{\frac{1}{2}} - x = 0,$$

$$\therefore 2x^{\frac{1}{2}}y^{\frac{1}{2}} = x - 2, \quad \therefore y^{\frac{1}{2}} = \frac{x-2}{2x^{\frac{1}{2}}},$$

$$\therefore y = \frac{(x-2)^2}{4x}, \ \therefore 64y = \frac{16(x-2)^2}{x}. \quad \text{Square (2).}$$

$$64 + 16x + x^2 = 64y + 128,$$

$$\therefore x^2 + 16x - 64 = \frac{16(x-2)^2}{x},$$

$$x^3 + 16x^2 - 64x = 16x^2 - 64x + 64,$$

$$x^3 = 64, \quad \therefore x = 4,$$

$$y = \frac{(4-2)^2}{16} = \frac{2^2}{16} = \frac{4}{16} = \frac{1}{4}.$$

11. $\left.\begin{aligned} x^4 &= 3x + 2y \\ y^4 &= 2x + 3y \end{aligned}\right\}$ Add and subtract.

$x^4 + y^4 = 5(x+y)$, (1), $\quad x^4 - y^4 = x - y$, (2),

(2) $(x^2 + y^2)(x^2 - y^2) = x - y$, Divide by $x - y$.

$$(x^2 + y^2)(x + y) = 1 \ldots\ldots (3), \ \therefore x + y = \frac{1}{x^2 + y},$$

(1) $x^4 + y^4 = \dfrac{5}{x^2 + y^2}$, $\therefore (x^4 + y^4)(x^2 + y^2) = 5 \ldots (4)$,

$\dfrac{(x^2 + y^2)^2 (x+y)^2}{(x^4 + y^4)(x^2 + y^2)} = \dfrac{1}{5}$, by squaring (3) and dividing by (4),

$$\therefore \frac{(x^2 + y^2)(x+y)^2}{x^4 + y^4} = \frac{1}{5}$$

or $\dfrac{x^4 + 2x^2y^2 + y^4 + 2x^3y + 2xy^3}{x^4 + y^4} = \dfrac{1}{5}$

Clearing this equation, and dividing by $4x^2y^2$, we have

$$\frac{x^2}{y^2}+\frac{5}{2}+\frac{y^2}{x^2}+\frac{5}{2}\cdot\frac{x}{y}+\frac{5}{2}\cdot\frac{y}{x}=0. \quad \text{Subtract } \frac{1}{2}.$$

$$\frac{x^2}{y^2}+2+\frac{y^2}{x^2}+\frac{5}{2}\left(\frac{x}{y}+\frac{y}{x}\right)=-\tfrac{1}{2},$$

$$\text{or } \left(\frac{x}{y}+\frac{y}{x}\right)^2+\frac{5}{2}\left(\frac{x}{y}+\frac{y}{x}\right)=-\tfrac{1}{2}.$$

Completing the square, and extracting the root, we have

$$\frac{x}{y}+\frac{y}{x}=\frac{-5\pm\sqrt{17}}{4}. \quad \text{Multiply by } \frac{x}{y}, \text{ and transpose.}$$

$$\frac{x^2}{y^2}+\frac{5\mp\sqrt{17}}{4}\cdot\frac{x}{y}=-1. \quad \text{Complete the square, and extract the root.}$$

$$\frac{x}{y}=-\frac{5\mp\sqrt{17}\mp\sqrt{(-22\mp10\sqrt{17})}}{8}.$$

(2) $(x^2+y^2)(x+y)(x-y)-(x-y)=0,$

$$\{(x^2+y^2)(x+y)-1\}(x-y)=0,$$

$$\therefore x-y=0, \quad \therefore x=y.$$

It is evident from the last part of this solution, that any factor common to all the terms of an equation is equal to 0.

Again, $x^3=3x+2x=5x, \quad \therefore x^2=5, \quad \therefore x=\sqrt[3]{5}.$

When the *sum of the indices* of the unknown quantities is the same in every term, the equations are called *homogeneous*, and they may be solved, when no simpler solution presents itself, by assuming one of them to be equal to some unknown multiple of the other, as in the following example.

12. $\left.\begin{array}{l} x^2+y^2=34 \\ x^2-xy=10 \end{array}\right\}$ Assume $x=vy$, then $x^2=v^2y^2$, and $xy=vy^2$,

$$\therefore v^2y^2+y^2=34, \quad (v^2+1)y^2=34, \quad \therefore y^2=\frac{34}{v^2+1},$$

$$v^2y^2-vy^2=10, \quad (v^2-v)y^2=10, \quad \therefore y^2=\frac{10}{v^2-v}.$$

$$\text{Hence, } \frac{34}{v^2+1} = \frac{10}{v^2-v}, \qquad \text{or } \frac{17}{v^2+1} = \frac{5}{v^2-v},$$

$$17v^2 - 17v = 5v^2 + 5, \qquad 12v^2 - 17v = 5,$$

$\therefore v^2 - \frac{17}{12}v = \frac{5}{12}$, and solving this quadratic, we have $v = \frac{5}{3}$,

$$\therefore y^2 = \frac{10}{v^2 - v} = \frac{10}{\frac{25}{9} - \frac{5}{3}} = \frac{90}{25-15} = \frac{90}{10} = 9,$$

$$\therefore y = 3, \text{ and } x = vy = \frac{5}{3} \times 3 = 5.$$

13. $\left.\begin{aligned} x^2 + xy + y^2 &= 37 \\ x^2 + xz + z^2 &= 28 \\ y^2 + yz + z^2 &= 19 \end{aligned}\right\} \begin{aligned} &\dots (A). \\ &\dots (B). \\ &\dots (C). \end{aligned}$ The method adopted in the last example would apply to this; we will, however, give another solution.

$$\left.\begin{aligned} A - B, \quad & y^2 - z^2 + (y-z)x = 9 \\ B - C, \quad & x^2 - y^2 + (x-y)z = 9 \end{aligned}\right\}, \text{ or}$$

$$\left.\begin{aligned} y + z + x &= \frac{9}{y-z} \\ x + y + z &= \frac{9}{x-y} \end{aligned}\right\} \dots\dots (D).$$

$$\therefore \frac{9}{y-z} = \frac{9}{x-y}, \quad \text{or } y - z = x - y,$$

$$\therefore x + z = 2y, \quad \text{Substitute in (D).}$$

$$3y = \frac{9}{y-z}, \quad y = \frac{3}{y-z}, \quad y^2 - yz = 3,$$

$$\left.\begin{aligned} yz = y^2 - 3, \quad \therefore z &= \frac{y^2-3}{y} \\ z^2 &= \frac{y^4 - 6y^2 + 9}{y^2} \end{aligned}\right\} \text{ Substitute in (C).}$$

$$y^2 + y^2 - 3 + \frac{y^4 - 6y^2 + 9}{y^2} = 19,$$

$$2y^2 + \frac{y^4 - 6y^2 + 9}{y^2} = 22,$$

$$2y^4 + y^4 - 6y^2 + 9 = 22y^2,$$

$$3y^4 - 28y^2 = -9,$$

$y^4 - \frac{28}{3}y^2 = -3$, and, solving this quadratic, we have

$$y = \pm 3, \quad \therefore z = y - \frac{3}{y} = \pm 3 - \frac{3}{\pm 3} = \pm 2,$$

$$\text{and } x = 2y - z = \pm 6 \mp 2 = \pm 4.$$

Examples.

1. $\left.\begin{array}{l} x^2 + y^2 = 20. \\ x^2 - y^2 = 12. \end{array}\right\}$ Ans. $x = 4.$ $y = 2.$

2. $\left.\begin{array}{l} x + y = 6. \\ x^2 + y^2 = 26. \end{array}\right\}$ Ans. $x = 5.$ $y = 1.$

3. $\left.\begin{array}{l} x^2 + y^2 = 10. \\ x - y = 2. \end{array}\right\}$ Ans. $x = 3.$ $y = 1.$

4. $\left.\begin{array}{l} x^2 + y^2 = 25. \\ x + y = 1. \end{array}\right\}$ Ans. $x = 4$, or $-3.$ $y = -3$, or $4.$

5. $\left.\begin{array}{l} x^2 - y^2 = 16. \\ x + y = 8. \end{array}\right\}$ Ans. $x = 5.$ $y = 3.$

6. $\left.\begin{array}{l} x - y = 1. \\ x^3 - y^3 = 19. \end{array}\right\}$ Ans. $x = 3$, or $-2.$ $y = 2$, or -3

7. $\left.\begin{array}{l} x^3 + y^3 = 189. \\ x^2y + xy^2 = 180. \end{array}\right\}$ Ans. $x = 5$, or $4.$ $y = 4$, or $5.$

8. $\left.\begin{array}{l} 10x + y = 3xy. \\ y - x = 2. \end{array}\right\}$ Ans. $x = 2$, or $-\frac{1}{3}.$ $y = 4$, or $-\frac{5}{3}.$

9. $\left.\begin{array}{l} x^2 + y^2 + x + y = 18. \\ 2xy = 12. \end{array}\right\}$ Ans. $x = 3$, or 2, or $-3 \pm \sqrt{3}.$ $y = 2$, or 3, or $-3 \mp \sqrt{3}.$

10. $\left.\begin{array}{l} x^2y^4 + y^2 = 19. \\ xy^4 + y = 4. \end{array}\right\}$ Ans. $x = 3.$ $y = 1.$

11. $\left.\begin{array}{l}\dfrac{x+y}{x-y}=a^2 \\ x^2-y^2=b^2\end{array}\right\}$ Ans $x=\dfrac{b}{2}(a^2+1)$.
$x=\dfrac{b}{2}(a^2-1)$.

12. $9x^2=4y^2$, $3xy+2x+y=485$.
Ans. $x=10$, or $-10\frac{7}{9}$. $y=15$, or $-16\frac{1}{6}$.

13. $\left.\begin{array}{r}x^2+y^2-x-y=78. \\ xy+x+y=39.\end{array}\right\}$ Ans. $x=\dfrac{-13\pm\sqrt{-39}}{2}$
$y=\dfrac{-13\mp\sqrt{+39}}{2}$.

14. $\left.\begin{array}{l}\dfrac{1}{y}-\dfrac{1}{x}=\dfrac{1}{4}. \\ x^2y-xy^2=16.\end{array}\right\}$ Ans. $x=4$, or -2.
$y=2$, or -4.

15. $\left.\begin{array}{l}x^2+xy=a^2+ab. \\ y^2+yx=b^2+ab.\end{array}\right\}$ Ans. $x=a$.
$y=b$.

16. $\left.\begin{array}{l}12xy=5x+12y. \\ y^2-x^2=1.\end{array}\right\}$ Ans. $x=1\frac{1}{3}$.
$y=1\frac{2}{3}$.

17. $\left.\begin{array}{l}2y+3x=8. \\ 3y^2+2x^2=11.\end{array}\right\}$ Ans. $x=2$, $2\frac{4}{35}$.
$y=1$, $\frac{29}{35}$.

18. $\left.\begin{array}{l}y-x=2. \\ 3xy=10x+y\end{array}\right\}$ Ans. $x=2$, $-\frac{1}{3}$.
$y=4$, $1\frac{2}{3}$.

19. $\left.\begin{array}{l}x+y+\sqrt{(x+y)}=12. \\ x^3+y^3=189.\end{array}\right\}$ Ans. $x=5$, or 4.
$y=4$, or 5.

20. $\left.\begin{array}{r}4xy=96-x^2y^2. \\ x+y=6.\end{array}\right\}$ Ans. $x=4$, or 2, or $3\pm\sqrt{21}$.
$y=2$, or 4, or $3\mp\sqrt{21}$.

21. $\left.\begin{array}{l}\dfrac{x^2+y^2}{xy}=\dfrac{13}{6} \\ x^3y+xy^3=x^2y^2+42.\end{array}\right\}$ Ans. $x=3$.
$y=2$

22. $\left.\begin{array}{l}x^2+y^2+4x-6y=13. \\ xy-3x+2y=11.\end{array}\right\}$ Ans. $x=3, -1, -3, -7$
$y=4, 8, -2, 2$.

23 $\left.\begin{array}{l}(x^2+y^2)x^2y^2=3600. \\ x^2y+xy^2=84.\end{array}\right\}$ Ans. $x=3$, or 4.
$y=4$, or 3.

24. $\left.\begin{array}{l} x^4 + y^4 = 2657. \\ x + y = 11. \end{array}\right\}$ Ans. $x = 7$, or 4. $y = 4$, or 7.

25. $\left.\begin{array}{l} 2x + 2y - x^2 - y^2 + 2 = 0. \\ xy = 3. \end{array}\right\}$

Ans. $x = 3$, or 1, or $-(1 \pm \sqrt{-2})$.
$y = 1$, or 3, or $-(1 \mp \sqrt{-2})$.

26. $\left.\begin{array}{l} x^2 + y^2 = 61. \\ x^2 - xy = 6. \end{array}\right\}$ Ans. $x = 6$. $y = 5$.

27. $\left.\begin{array}{l} x + y = 10. \\ x^5 + y^5 = 17050. \end{array}\right\}$ Ans. $x = 7$, or 3. $y = 3$, or 7.

28. $\left.\begin{array}{l} x^2 + xy + y^2 = 21. \\ x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 3. \end{array}\right\}$ Ans. $x = 1$ $y = 4$.

29. $\left.\begin{array}{l} x^2y^2 + 12xy = 9x^2 + 4y^2. \\ x^2 + 4x + y^2 = 6y + 24. \end{array}\right\}$ Ans. $x = 4$, or -3. $y = 2$, or 9

30. $\left.\begin{array}{l} x - y = 2. \\ x^4 + y^4 = 272. \end{array}\right\}$ $x = 4,\ -2$, or $\sqrt{-15} + 1$. $y = 2,\ -4$, or $\sqrt{-15} - 1$

31. $\left.\begin{array}{l} x^2 + 2xy + y + 3x = 73. \\ y^2 + x + 3y = 44. \end{array}\right\}$ Ans. $x = 4$, or 16. $y = 5$, or -7.

32. $\left.\begin{array}{l} (x^2 + y^2)(x + y) = 120. \\ (x - y)(x^2 - y^2) = 24. \end{array}\right\}$ Ans. $x = 4$. $y = 2$.

33. $\left.\begin{array}{l} xy = 6. \\ 3x^2 - 7y^2 + 1 = 0. \end{array}\right\}$ Ans. $x = \pm 3$ $y = \pm 2$.

34. $\left.\begin{array}{l} 2x^{-\frac{1}{2}}y^{-\frac{1}{2}} = 1 + x^{\frac{1}{2}}y^{\frac{1}{2}} \\ x - y = \sqrt{(a - x + y)} - \sqrt{(a + x - y)} \end{array}\right\}$ $x = y = 1$.

35. $x - y = 3$, $x^3 + y^3 = 19(x + y)$.

Ans. $x = 5,\ -2$, $y = 2,\ -5$

36 $\left.\begin{array}{l} x - 2\sqrt{xy} + y - \sqrt{x} + \sqrt{y} = 0. \\ \sqrt{x} + \sqrt{y} = 5. \end{array}\right\}$ $x = 9$, or $\frac{25}{4}$. $y = 4$, or $\frac{25}{4}$

37. $\left.\begin{array}{l}\dfrac{x^2}{y^2} + \dfrac{4x}{y} = 9\frac{4}{9}. \\ x - y = 2\end{array}\right\}$ Ans. $x = 5$, $y = 3$.

38. $\left.\begin{array}{l}x^4 - 2x^2y + y^2 = 49, \\ x^4 - 2x^2y^2 + y^4 - x^2 + y^2 = 20.\end{array}\right\}$
Ans. $x = \pm 3$, or $\pm\sqrt{6}$ $y = 2$, or -1.

39. $\left.\begin{array}{l}x^n + y^n = a \\ xy = b\end{array}\right\}$ Ans. $x = \sqrt[n]{\frac{1}{2}\{\sqrt{(a^2 - 4b^n)} + a\}}$.
$y = \dfrac{b}{\sqrt[n]{\frac{1}{2}\{\sqrt{(a^2 - 4b^n)} + a\}}}$.

40. $\left.\begin{array}{l}\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} - \dfrac{3}{2}\left(\dfrac{x}{y} + \dfrac{y}{x}\right) = 2\frac{3}{50}. \\ 4x - 5y = 10.\end{array}\right\}$ Ans. $x = 5$. $y = 2$.

41. $x^2 - xy = 48y$, $xy - y^2 = 3x$.
Ans. $x = 16$, or $-9\frac{3}{5}$, $y = 4$, or $2\frac{2}{5}$.

42. $\left.\begin{array}{l}x^2 + y^2 - 15(x + y) = -70. \\ 3xy + 31(x + y) = 210.\end{array}\right\}$ Ans $x = 2$. $y = 4$.

43. $\left.\begin{array}{l}\sqrt{y} : \sqrt{x} :: \sqrt{x} + 3 : \sqrt{x} + 1. \\ \sqrt{xy} + 2\sqrt{y} = 3x + 3\sqrt{x}.\end{array}\right\}$ $x = 1$. $y = 4$.

44. $\left.\begin{array}{l}\dfrac{x}{y} - \dfrac{y}{x} = \dfrac{11}{30}. \\ x^2 + xy = 66.\end{array}\right\}$ Ans. $x = 6$. $y = 5$

45. $\left.\begin{array}{l}3x^2 = 2xy + 24. \\ y^2 = xy - 3.\end{array}\right\}$ Ans. $x = 4$. $y = 3$.

46. $x^2 - 2xy + 3y^2 = 9$, $x^2 - 4xy + 5y^2 = 5$
Ans. $x = \pm 3$, and $\pm\dfrac{5}{\sqrt{2}}$, $y = \pm 2$, and $\pm\dfrac{1}{\sqrt{2}}$

47. $\left.\begin{array}{l} 2x^2 - 3xy + y^2 = 4. \\ x^2 - 2xy + 3y^2 = 9. \end{array}\right\}$ Ans. $x = \pm 3$, $y = \pm 2$.

48. $\left.\begin{array}{l} x^{\frac{2}{3}} + y^{\frac{2}{3}} = 3x. \\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = x. \end{array}\right\}$ Ans. $x = 4$, or 1. $y = 8$

49. $\left.\begin{array}{l} \sqrt{\dfrac{3x}{x+y}} + \sqrt{\dfrac{x+y}{3x}} = 2. \\ xy - (x+y) = 54. \end{array}\right\}$ $x = 6$, or $-\frac{9}{2}$. $y = 12$, or -9.

50. $\left.\begin{array}{l} (x + y) = 3(x - y)^{\frac{1}{3}}. \\ (x^3 + y^3)(x + y) = 27. \end{array}\right\}$ Ans. $x = 2$. $y = 1$.

51. $\left.\begin{array}{l} \dfrac{x^2 + xy + y^2}{x^2 - xy + y^2} = 2\frac{1}{3}. \\ \dfrac{1}{x} + \dfrac{1}{y} = 1\frac{1}{2}. \end{array}\right\}$ Ans. $x = 2$, or 1. $y = 1$, or 2.

52. $\left.\begin{array}{l} 2x^2 - 2xy = 3y. \\ 3xy - 3y^2 = 2x. \end{array}\right\}$ Ans. $x = 3$. $y = 2$.

53. $\left.\begin{array}{l} (x + y)(x - y)^2 = 32. \\ x^2 - y^2 - x - y = 8. \end{array}\right\}$ Ans. $x = 5$. $y = 3$.

54. $\left.\begin{array}{l} 4xy^2 - x^2y^2 = \dfrac{y^6}{4} - 4. \\ x^3 - xy(x - y) = 3. \end{array}\right\}$ Ans. $x = 1$. $y = 2$.

55. $\left.\begin{array}{l} (x + y)^{\frac{1}{2}} - (x - y)^{\frac{1}{2}} = a. \\ (x + y)^{\frac{1}{4}} + (x - y)^{\frac{1}{4}} = b. \end{array}\right\}$

Ans. $x = \dfrac{a^2}{4} + \left\{\dfrac{1}{4}\left(b^2 + \dfrac{a^2}{b^2}\right)\right\}^2$. $y = \dfrac{a}{4}\left(b^2 + \dfrac{a^2}{b^2}\right)$

56. $\left.\begin{array}{l} (x^2 + y^2)(x + y) = 2xy. \\ (x^4 - y^4)(x^2 + y^2) = x^2y^2. \end{array}\right\}$ Ans. $x = \frac{75}{136}$. $y = \frac{45}{136}$

57. $\left.\begin{aligned}(x^2 + y^2)\frac{y}{x} &= \frac{26}{3}.\\ (x^2 - y^2)\frac{x}{y} &= \frac{15}{2}.\end{aligned}\right\}$ Ans. $x = 3$. $y = 2$.

58. $\left.\begin{aligned}(x^2 + y^2) - (x^4 + y^4) &= \frac{63}{256}.\\ (x + y)^2 + (xy - 2)xy &= \frac{21}{64}.\end{aligned}\right\}$ Ans. $x = \frac{1}{2}$. $y = \frac{1}{4}$.

59. $\left.\begin{aligned}(x - 2)y + x - 2y^2 &= (y^2 - 1)\sqrt{xy}.\\ \frac{\sqrt{xy} - 12}{xy - 18} &= \frac{xy}{4}.\end{aligned}\right\}$ Ans. $x = 8$. $y = 2$.

60. $\left.\begin{aligned}(x^4 - y^4)(x^2 - y^2) &= 45x^2y^2.\\ (x^2 + y^2)(x + y) &= 15xy.\end{aligned}\right\}$ Ans. $x = 4$. $y = 2$.

61. $\left.\begin{aligned}\sqrt[4]{(x + y)} + \sqrt[4]{(x - y)} &= y\\ xy + \sqrt{(x^2y^2 - y^4)} &= \sqrt{(x + y)} + \sqrt{(x - y)}.\end{aligned}\right\}$

Ans. $x = \sqrt[3]{2}$, $y = \sqrt[3]{2}$.

62. $\left.\begin{aligned}8\sqrt{(y + 2)} &= x + 8.\\ x^{\frac{1}{2}} - y^{\frac{1}{2}} &= (y + 2)^{\frac{1}{2}}.\end{aligned}\right\}$ Ans. $x = 4$. $y = \frac{1}{4}$.

63. $\left.\begin{aligned}x^2 - xy + y^2 &= \frac{91}{x^2 + y^2}.\\ x^2 + xy + y^2 &= \frac{133}{x^2 - xy + y^2}.\end{aligned}\right\}$ Ans. $x = 3$. $y = 2$.

64. $\left.\begin{aligned}\sqrt{(x^2 + \sqrt[3]{x^4y^2})} + \sqrt{(y^2 + \sqrt[3]{x^2y^4})} &= a.\\ x + y + 3\sqrt[3]{bxy} &= b.\end{aligned}\right\}$

Ans. $x = \frac{1}{8}\{b^{\frac{1}{3}} + \sqrt{(2a^{\frac{2}{3}} - b^{\frac{2}{3}})}\}^3$

$y = \frac{1}{8}\{b^{\frac{1}{3}} - \sqrt{(2a^{\frac{2}{3}} - b^{\frac{2}{3}})}\}^3$

65. $\left.\begin{aligned}x^2 + y^2 &= 3xy.\\ x^5 + y^5 &= 2.\end{aligned}\right\}$ Ans. $x = \frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5} + 1)$. $y = \frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5} - 1)$.

66. $$\left.\begin{aligned}&(x+y)^3 = 64(x-y).\\&(x^3+y^3)(x+y) = 76.\end{aligned}\right\}$$ Ans. $x = \frac{5}{2}$. $y = \frac{3}{2}$

67. $$\left.\begin{aligned}&x^4 + y^4 = 1 + 2xy + 3x^2y^2.\\&x^3 + y^3 = 2y^2x + 2y^2 + x + 1.\end{aligned}\right\}$$ Ans. $x = 2$. $y = 1$.

68. $$\left.\begin{aligned}&(x^2 + xy + y^2)(x^2 - xy + y^2) = 481.\\&(x^2 + y^2)^2 - (x^2 + y^2)xy = 325.\end{aligned}\right\}$$ Ans. $x = 3$. $y = 4$.

69. $$\left.\begin{aligned}&x - 2\sqrt{2ay - y^2} = 5\sqrt{ay}.\\&x^2 - 8\sqrt{ay}\,.\,x = 2ay - 9y^2.\end{aligned}\right\}$$ Ans. $x = 7a$. $y = a$.

70. $$\left.\begin{aligned}&x^6 + y^6 = \frac{1}{\sqrt{2}}.\\&x^4 + y^4 = 1\end{aligned}\right\}$$ $x = y = \frac{1}{\sqrt[4]{2}}$.

71. $$\left.\begin{aligned}&x^7 - y^7 = 127.\\&x - y = 1.\end{aligned}\right\}$$ Ans. $x = 2$. $y = 1$.

72. $$\left.\begin{aligned}&x + y + z = 3a.\\&xy + xz + yz = 3a^2.\\&xyz = a^3.\end{aligned}\right\}$$ Ans. $x = a$. $y = a$. $z = a$.

73. $$\left.\begin{aligned}&xy = x + y.\\&xz = 2(x + z).\\&yz = 3(y + z).\end{aligned}\right\}$$ Ans. $x = 1\frac{5}{7}$. $y = 2\frac{2}{3}$. $z = -12$.

74. $$\left.\begin{aligned}&x^2 + y^2 + z^2 = 14.\\&x^2 + y = 3.\\&z^2 + y = 11.\end{aligned}\right\}$$ Ans. $x = \pm 1$. $y = 2$. $z = \pm 3$.

75. $$\left.\begin{aligned}&xy + z = 5.\\&xyz = 4.\\&2(x^2 - y) = (y^2 - x)^2.\end{aligned}\right\}$$ Ans. $x = 2,\ -2\sqrt[3]{4} \pm 4$. $y = 2,\ -\sqrt[3]{2} \pm 1$. $z = 1,\ 4$.

76. $$\left.\begin{aligned}&x^2 + xy + y^2 = 13.\\&y^2 + yz + z^2 = 49.\\&x^2 + xz + z^2 = 31.\end{aligned}\right\}$$ Ans. $x = \pm 1$. $y = \pm 3$. $z = \pm 5$.

77. $$\left.\begin{aligned}&x^{-2}y^{-1}z = 1\tfrac{1}{2}.\\&x^{-1}yz^2 = 18.\\&xy^2z^3 = 108.\end{aligned}\right\}$$ Ans. $x = \pm 1$. $y = \pm 2$. $z = \pm 3$.

78. $\left.\begin{aligned}(x+\sqrt{y})^2 &= y^2 - 4xz.\\ x+\sqrt{y} &= \frac{x+z}{2}\\ 4\sqrt{y} &= x+z.\end{aligned}\right\}$ Ans. $x = 4.$ $y = 16.$ $z = 12.$

Problems.

1. A sets out from London to York B from York to London; A arrives in York 9 hours, and B in London 16 hours, after they had met. In what time did each perform the journey?

Let x = number of hours each was on the road before they met,

$$\text{then } x + 9 = \text{A's whole time},$$
$$x + 16 = \text{B's},$$

hours. hours. journey.

$$x + 9 : x :: 1 : \frac{x}{x+9} = \text{the part A goes before they met,}$$

$$x + 16 : x :: 1 : \frac{x}{x+16} = \quad \text{"} \quad \text{B} \quad \text{"}$$

Hence, $\dfrac{x}{x+9} + \dfrac{x}{x+16} = 1$, (the whole journey)

$$x^2 + 16x + x^2 + 9x = x^2 + 25x + 144,$$
$$\therefore x^2 = 144, \quad x = 12,$$
$$\therefore x + 9 = 21 = \text{A's time in hours},$$
$$x + 16 = 28 = \text{B's} \quad \text{"} .$$

Or thus: let x be A's whole time,

y " B's "

$$x : 9 :: 1 : \frac{9}{x}, \text{ A's part of distance after they met,}$$

$$x : 16 :: 1 : \frac{16}{y}, \text{ B's,} \quad \text{"}$$

Hence, $\dfrac{9}{x} + \dfrac{16}{y} = 1$, $\therefore 9y + 16x = xy.$

But $x = y - 7$, $\therefore$ by substitution,

$$9y + 16y - 112 = y^2 - 7y,$$
$$y^2 - 32y = -112,$$

$\therefore y = 28$ hours $=$ B's time, $x = y - 7 = 21$ hours $=$ A's time.

2. A and B set out at the same time, A from C to go to D, and B from D to go to C; they meet on the road, when it appears that A has travelled 30 miles more than B, and that, at the rate he is travelling, he will reach D in 4 days, and that B will arrive at C in 9 days. Find the distance of C from D.

Let $x =$ A's distance before they met,

$y =$ B's ,,

$\therefore\ x = y + 30.$

Now $y =$ miles A goes in 4 days,

$y : x :: 4 : \dfrac{4x}{y} =$ N°. of days A is travelling before they meet.

Again, $x =$ miles B goes in 9 days,

$x : y :: 9 : \dfrac{9y}{x} =$ N°. of days B is travelling before they meet

$$\text{Hence, } \frac{4x}{y} = \frac{9y}{x};$$

$$\therefore\ 4x^2 = 9y^2, \quad \therefore\ 2x = 3y, \quad \therefore\ x = \frac{3y}{2}.$$

$$\text{Hence, } \frac{3y}{2} = y + 30, \quad \therefore\ y = 60, \text{ and } x = 90,$$

$$\therefore\ x + y = 150 \text{ miles} = \text{the distance. Ans.}$$

3. What are those two numbers whose difference is 2, and whose product multiplied by their sum is 12?

Let $x =$ the greater, and $y =$ the less,

$$\text{then } x - y = 2, \qquad \text{or } y = x - 2,$$

$$\text{and } xy(x + y) = 12, \qquad \text{or } x^2y + xy^2 = 12,$$

$$x^2(x - 2) + x(x - 2)^2 = 12,$$

$$x^3 - 2x^2 + x^3 - 4x^2 + 4x = 12,$$

$$2x^3 - 6x^2 + 4x - 12 = 0,$$

$$2x^2(x - 3) + 4(x - 3) = 0,$$

$$\therefore\ x = 3, \ y = 1.$$

4. The sum of two numbers multiplied by the sum of their cubes is 112, and the cube of their sum is to their difference as 32 to 1; find the numbers.

Let x = the greater, and y = the less,

$$\text{then } (x+y)(x^3+y^3) = 112,$$
$$\text{and } (x+y)^3 : x-y :: 32 : 1.$$
$$(x+y)^4 : x^2-y^2 :: 32 : 1, \text{ or}$$
$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 32(x^2-y^2),$$
$$4x^4 + 4x^3y \qquad + 4xy^3 + 4y^4 = 448$$
$$3x^4 \qquad - 6x^2y^2 \qquad + 3y^4 = 448 - 32(x^2-y^2)$$
$$x^4 - 2x^2y^2 + y^4 + \frac{32}{3}(x^2-y^2) = \frac{448}{3},$$
$$(x^2-y^2)^2 + \frac{32}{3}(x^2-y^2) + \left(\frac{16}{3}\right)^2 = \frac{256}{9} + \frac{1344}{9} = \frac{1600}{9},$$
$$x^2 - y^2 + \frac{16}{3} = \frac{40}{3},\quad x^2-y^2 = \frac{24}{3} = 8,$$
$$\therefore (x+y)^4 = 32(x^2-y^2) = 256,$$
$$x+y=4,\quad x-y=2,$$
$$\therefore x = 3,\quad y = 1.$$

Problems producing Equations.

1. Required two numbers whose difference is 8, and product 128. Ans. $\pm$ 8, and $\pm$ 16.

2. There are two numbers whose sum is 40, and the sum of their squares is 818; find them. Ans. 23 and 17.

3. What magnitude is that which exceeds its reciprocal by 1? Ans. $\frac{1}{2}(1 \pm \sqrt{5})$.

4. A person bought as many sheep as cost him 60*l.*, and after reserving 15 out of the number, sold the remainder for 54*l.*, and gained 2*s.* a head by them; how many did he buy? Ans. 75.

5. A, B, and C together perform a piece of work in a certain time; A alone could have done it in 6 hours more, B alone in 15 hours more, and C alone in twice the time: how long did it occupy them? Ans. 3 hours.

6. In a circular grass plot, a walk, A B, cuts another, C D,

into two equal parts in the point E; A E = 25 feet, E B = $\frac{4}{5}$ of C D — 16 feet; required the length of C D. Ans. 40 feet.

7. A grazier bought a certain number of oxen for 240*l.*, and after losing 3, sold the remainder at 8*l.* a head more than they cost him, thus gaining 59*l.* by his bargain; what number did he buy? Ans. 16.

8. The reckoning of a party at a tavern was 3*l.* 12*s.*, but in consequence of two of them having no money, each of the rest paid 6*d.* more than he otherwise should have done; required their number. Ans. 18.

9. A person at play won twice as much as he began with, and then lost 16 shillings; after this he lost four-fifths of what remained, and then won as much as he began with, and counting his money found he had 80 shillings; what sum did he begin with? Ans. 52*s.*

10. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 10 yards, and it contains 3000 square yards; required the length and breadth. Ans. 60 and 50 yards.

11. Two partners, A and B, gained 18*l.* by trade; A's money was in trade 12 months, and he received for his principal and gain 26*l.*; also B's money, which was 30*l.*, was in trade 16 months; what money did A put into trade? Ans. 20*l.*

12. There are two square buildings that are paved with stones a foot square each, the side of one building exceeds that of the other by 12 feet, and both their pavements together contain 2120 stones; what are the lengths of them separately? Ans. 26 and 38 feet respectively.

13. A person by selling a horse for 56*l.*, gains as much per cent. as the horse cost him; what was its original price? Ans. 40*l.*

14. The difference of two numbers is 6, and the sum of their squares multiplied by their product is 4640; find them. Ans. 10 and 4.

15. A and B start at the same time to travel 150 miles, A travels 3 miles an hour faster than B, and finishes his journey 8⅓ hours before him; what is the rate of each? Ans. 9 and 6 miles per hour.

16. A man working for 10 hours, assisted by a boy who works for 6 hours, does a certain piece of work; if the man had worked for 6 hours and the boy for 10, only two thirds

of the work would have been done; how long will it take the man and the boy to do the work, supposing the man to work 5 hours longer than the boy?

Ans. man, $10\frac{1}{10}$ hours, boy, $5\frac{1}{10}$ hours.

17. A person rows 20 miles down a river and back again in 10 hours, and he finds that he can row two miles against the stream in the same time that he can row three miles with it; required the rate of the stream, and the times of his going and returning.

Ans. $\frac{5}{6}$ of a mile per hour, and 4 and 6 hours.

18. A body of men are just sufficient to form a hollow equilateral wedge, three deep, and if 597 be taken away, the remainder will form a hollow square four deep, the front of which contains one man more than the square root of the number contained in a front of the wedge; what is the number of men? Ans. 693.

19. Two merchants enter into partnership with 100*l.*; one has his money in business for three months, and the other for two months; and each receives 99*l.* for his capital and profit; find the contribution of each. Ans. 45*l.* and 55*l.*

20. Two detachments of infantry are ordered to a station distant 39 miles; they begin their march at the same time, but one party by travelling $\frac{1}{4}$ of a mile an hour more than the other, arrives one hour sooner; required the rates of marching. Ans. 3 and $3\frac{1}{4}$ miles an hour.

21. A vintner sold 7 dozen of sherry and 12 dozen of claret for 50*l.*; he sold 3 dozen more of sherry for 10*l.* than he did of claret for 6*l.*; required the price of each.

Ans. Sherry, 2*l.* per dozen; claret, 3*l.* per dozen.

22. The number of men in both fronts of two columns of troops, A and B, when each consisted of as many ranks as it had men in front, was 84; but when the columns changed ground, and A was drawn up with the front B had, and B with the front A had, the number of ranks in both columns was 91; required the number of men in each column.

Ans. 2304 and 1296

23. A and B lay out some money in speculation; A disposes of his bargain for 11*l* and gains as much per cent. as B lays out; B's gain is 36*l.*, and it appears that A gains four times as much per cent. as B; required the capital of each.

A's capital, 5*l.*, and B's, 120*l.*

24. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men, and by this movement the detachment was drawn up in five lines; required the number of men. Ans. 4550 men.

25. In a mixture of wine and cider, one-half, together with 25 gallons, was wine, and the cider was less than one-third part of the mixture by 5 gallons; how many gallons of each did it contain? Ans. 85 of wine, and 35 of cider.

26. The difference between the hypothenuse and base of a right-angled triangle is $= 6$, and the difference between the hypothenuse and the perpendicular is $= 3$; what are the sides? Ans. 15, 12, and 9.

27. There is a number, consisting of two digits; and being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10; required the number. Ans. 23.

28. A and B gained by trading 100*l.*; half of A's stock was less than B's by 100*l.* and A's gain was three-twentieths of B's stock; what did each put into stock, and what are the respective shares of the gain?

Ans. A's stock was 600*l.*, and B's 400*l.*; A's gain was 60*l.*, B's 40*l.*

29. From two places, at the distance of 320 miles, two persons, A and B, set out at the same time to meet each other; A travelled 8 miles a day more than B, and the number of days until they met was equal to half the number of miles B went in a day; how many miles did each travel per day, and how far did each travel?

Ans. A went 24, B 16 miles per day; A went 192, B 128 miles.

30. There are two rectangular vats, the greater of which contains 20 solid feet more than the other. Their capacities are in the ratio of 4 to 5, and their bases are squares, a side of each of which is equal to the depth of the other; what are the depths? Ans. 5 feet, and 4 feet.

31. Three persons divide a certain sum of money amongst them in the following manner: A takes the nth part of the whole, together with $\frac{a}{n}$£, B takes the nth part of the remain-

der, together with $\frac{a}{n}$ £, C takes the nth part of that which now remains, together with $\frac{a}{n}$ £, after which nothing remains; find the sum of money.

Ans. $\frac{3n^2 - 3n + 1}{(n-1)^3} a$ £.

32. The distance between two places is a, and on the first day $\frac{1}{m}$th of the journey is performed, on the second day $\frac{1}{n}$th of the remainder, then $\frac{1}{m}$th and $\frac{1}{n}$th of the remainders alternately on succeeding days; find the distance gone over in $2p$ days.

Ans. $a\left\{1 - \left(1 - \frac{1}{m}\right) \cdot \left(1 - \frac{1}{n}\right)^p\right\}$.

CHAPTER VII.

INEQUALITIES, RATIO, PROPORTION, AND VARIATION.

XVIII. *Inequalities* are indicated by the sign $>$ *greater than*, or $<$ *less than*; thus $5 > 3$, $4 < 6$, $a > b$, are inequalities. They may be treated in the same manner as equations, excepting that when *all* the terms have their signs changed, the sign $>$ must be changed to $<$, and the sign $<$ to $>$; for whenever a is $> b$, $-a$ is necessarily $< -b$; for instance, $5 > 3$. but $-5 < -3$.

Examples.

1. Let m and n be any two *unequal* quantities, then

$$m^2 + n^2 > 2mn.$$

For ∵ no square quantity can be negative,

$\therefore (m-n)^2$ or $m^2 + n^2 - 2mn$ is positive;

∴ the positive part of this expression is $>$ the negative part; that is, $m^2 + n^2 > 2mn$

2. Show that $\sqrt{11} + \sqrt{7}$ is greater than $\sqrt{19} + \sqrt{2}$

$$\sqrt{11} + \sqrt{7} > \text{ or } < \sqrt{19} + \sqrt{2}$$

according as $(\sqrt{11} + \sqrt{7})^2 > \text{ or } < (\sqrt{19} + \sqrt{2})^2$,

or, $\quad 18 + 2\sqrt{77} > \text{ or } < 21 + 2\sqrt{38}$,

or, $\quad 2\sqrt{77} > \text{ or } < 3 + 2\sqrt{38}$, by subtracting 18 from each,

or, $308 > \text{ or } < 161 + 6\sqrt{38}$; by squaring each,

or, $147 > \text{ or } < 6\sqrt{38}$, by subtracting 161 from each.

Now it is evident that 147 is greater than $6\sqrt{38}$,

$\therefore \sqrt{11} + \sqrt{7}$ is greater than $\sqrt{19} + \sqrt{2}$.

3. Show that every fraction + its reciprocal is > 2.

Let $\frac{m}{n}$ be the fraction. then $\frac{n}{m}$ is its reciprocal.

$$\text{Now } \frac{m}{n} + \frac{n}{m} > \text{ or } < 2$$

$$\text{according as } \left(\frac{m}{n} + \frac{n}{m}\right)^2 > \text{ or } < 2^2,$$

$$\text{or, } \frac{m^2}{n^2} + 2 + \frac{n^2}{m^2} > \text{ or } < 4,$$

$$\text{or, } \frac{m^2}{n^2} + \frac{n^2}{m^2} > \text{ or } < 2.$$

$$\text{But } \because 2\frac{m}{n}\cdot\frac{n}{m} = 2, \quad \therefore \frac{m^2}{n^2} + \frac{n^2}{m^2} > 2 \text{ (by } Ex.\ 1)$$

$$\text{Hence } \frac{m}{n} + \frac{n}{m} > 2.$$

4. If $y = 1 + 4x - x^2$, what value of x makes y the greatest possible?

$$x^2 - 4x = 1 - y,$$
$$x^2 - 4x + 4 = 5 - y, \quad \therefore x - 2 = \sqrt{5 - y},$$
$$\therefore x = 2 \pm \sqrt{5 - y}.$$

Ans. x must $= 2$, and then y will $= 5$; for the value of x above found, namely, $2 \pm \sqrt{5 - y}$, becomes impossible when $y > 5$; and when y becomes $= 5$, the expression $\pm \sqrt{5 - y}$ vanishes, and $x = 2$. [For additional Examples, see Appendix.]

5. Which is greater, $\sqrt{7} + \sqrt{10}$, or $\sqrt{3} + \sqrt{19}$?
Ans. $\sqrt{3} + \sqrt{19}$.

6. If $4x - 7 < 2x + 3$, and $3x + 1 > 13 - x$, find an integral value of x. Ans. $x = 4$.

7. Show that $\frac{a + b}{2} > \frac{2ab}{a + b}$; and $a^5 + b^5 > a^4b + ab^4$.

8. Show that $\frac{n^2 - n + 1}{n^2 + n + 1}$ lies between 3 and $\frac{1}{3}$, for all real values of n.

RATIO.

[Although Ratio and Proportion have been occasionally used already, yet the *general* theory remains to be established.]

XIX. *Ratio* is the relation which one quantity bears to another with respect to magnitude; thus 12 is 3 times as great as 4, and the ratio of 12 to 4 is 3; this is expressed 12 : 4, the former 12 being called the *antecedent*, and the latter 4 the *consequent* of the ratio.

It is obvious that this ratio might be expressed by the fraction $\frac{12}{4}$, $\because \frac{12}{4} = 3$.

Similarly the ratio $12 : 36 = \frac{12}{36} = \frac{1}{3}$, and in general the value of the ratio of $a : b$ may be expressed by $\frac{a}{b}$.

The value of a ratio is not altered by multiplying or dividing both its terms by the same quantity.

$$\text{For } a : b = \frac{a}{b} = \frac{na}{nb}, \quad \therefore a : b = na : nb$$

A ratio is called a ratio *of greater inequality*, *of less inequality*, or *of equality*, according as the antecedent is *greater than*, *less than*, or *equal to*, the consequent.

A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.

If $a:b$ be the original ratio, and x be added to both its terms, the new ratio is $a+x:b+x$. Now $\frac{a+x}{b+x} >$ or $< \frac{a}{b}$ according as $\frac{ab+bx}{b(b+x)} >$ or $< \frac{ab+ax}{b(b+x)}$ by reduction to a common denominator,

$\therefore \frac{a+x}{b+x} >$ or $< \frac{a}{b}$ according as $ab+bx >$ or $< ab+ax$, according as $bx >$ or $< ax$, according as $b >$ or $< a$ (Art. XVIII.); that is,

$$\text{if } a > b,\ \frac{a+x}{b+x} < \frac{a}{b}, \text{ or the ratio is diminished;}$$

$$\text{if } a < b,\ \frac{a+x}{b+x} > \frac{a}{b}, \text{ or the ratio is increased.}$$

A ratio of greater inequality is increased, and a ratio of less inequality is diminished by subtracting the same quantity from both its terms.

If x, less than a and b, be subtracted from them both, the original ratio $= \frac{a}{b}$, and the new ratio $= \frac{a-x}{b-x}$;

$$\therefore \frac{a-x}{b-x} > \text{ or } < \frac{a}{b} \text{ as } \frac{ab-bx}{b(b-x)} > \text{ or } < \frac{ab-ax}{b(b-x)}$$

that is, as $ab-bx >$ or $< ab-ax$;

whence, adding $ax-bx$ to both members of the inequality, in order to avoid detached negative quantities,

$$\frac{a-x}{b-x} > \text{ or } < \frac{a}{b} \text{ according as}$$

$$ab - bx + ax + bx > \text{ or } < ab - ax + ax + bx;$$

as, $ab + ax >$ or $< ab + bx$; as $ax >$ or $< bx$: as $a >$ or $< b$, contrary to the last; that is.

$$\text{if } a > b, \frac{a-x}{b-x} > \frac{a}{b}, \text{ or the ratio is increased;}$$

$$\text{if } a < b, \frac{a-x}{b-x} < \frac{a}{b}, \text{ or the ratio is diminished.}$$

If the ratios $a : b$ and $c : d$ have their antecedents multiplied together, and also their consequents, the resulting ratio $ac : bd$ is said to be *compounded* of the two former.

The ratio $a^2 : b^2$ is called the *duplicate* ratio of a to b.

$\sqrt{a} : \sqrt{b}$	,,	*subduplicate.*	,,
$a^3 : b^3$	,,	*triplicate.*	,,
$\sqrt[3]{a} : \sqrt[3]{b}$	,,	*subtriplicate.*	,,
&c.		&c.	

PROPORTION.

XX. *Proportion* is the equality of ratios. Thus if the ratio $a : b$ be *equal* to the ratio $c : d$, that is, if $\frac{a}{b} = \frac{c}{d}$, the four quantities a, b, c, d are called *proportionals*, and $a : b :: c : d$, or $a : b = c : d$, is called a *proportion.*

Propositions.

1. If four quantities are proportionals, the product of the *extremes* is equal to the product of the *means*.

Let $a : b :: c : d$ be the proportion.

then $\frac{a}{b} = \frac{c}{d}$, and, multiplying both members of this equation by bd, we have $ad = bc$.

Cor. 1. If $a : b :: b : c$, then $ac = b^2$, and $\therefore b = \sqrt{ac}$, where b is a *mean proportional* to a and c. Hence a mean proportional between any two quantities is equal to the square root of their product.

Cor. 2. $\because ad = bc$, $\therefore a = \frac{bc}{d}$, $b = \frac{ad}{c}$, $c = \frac{ad}{b}$, $d = \frac{bc}{a}$;

Hence, if *three* terms of a proportion be given, the *fourth* may be found. The Rule of Three, in Arithmetic, is derived from this.

2. If the product of two quantities be equal to the product of two others, the four will constitute a proportion if the terms of one product be made the extremes, and the terms of the other the means.

Let $nx = my$, then, dividing by ny,

$$\frac{x}{y} = \frac{m}{n}, \quad \therefore x : y :: m : n.$$

3. If $a : b :: c : d$, and $c : d :: m : n$, then $a : b :: m : n$. (Euclid, Book V. Prop. 11.)

For $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{m}{n}$, $\quad \therefore \frac{a}{b} = \frac{m}{n}$,

$$\therefore a : b :: m : n.$$

4. If $a : b :: c : d$, then $b : a :: d : c$. (Euclid, Book V. Prop. B.)

For $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{b}{a} = \frac{d}{c}$, $\therefore b : a :: d : c$.

This operation is called *invertendo*.

5. If $a : b :: c : d$, then $a : c :: b : d$. (Euclid, Book V Prop. 16.)

For $\frac{a}{b} = \frac{c}{d}$, and, multiplying by $\frac{b}{c}$, we have

$\frac{a}{c} = \frac{b}{d}$, $\therefore a : c :: b : d$.

This is called *alternando*.

6. If $a : b :: c : d$, then $a + b : b :: c + d : d$. (Euclid, Book V. Prop. 18.)

For $\frac{a}{b} = \frac{c}{d}$, and, adding 1 to each side,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \qquad \text{or } \frac{a+b}{b} = \frac{c+d}{d},$$

$$\therefore a + b : b :: c + d : d.$$

This is called *componendo*.

7. If $a : b :: c : d$, then $a - b : b :: c - d : d$. (Euclid, Book V. Prop. 17.)

For $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$,

or $\frac{a-b}{b} = \frac{c-d}{d}$, $\therefore a - b : b :: c - d : d$.

This is called *dividendo*.

8. If $a : b :: c : d$, then $a - b : a :: c - d : c$,

and $a : a - b :: c : c - d$. (Euclid, Book V. Prop. E.)

For $\frac{a-b}{b} = \frac{c-d}{d}$, by Prop. 7,

and $\frac{b}{a} = \frac{d}{c}$, „ 4,

$\therefore \frac{a-b}{b} \times \frac{b}{a} = \frac{c-d}{d} \times \frac{d}{c}$, or $\frac{a-b}{a} = \frac{c-d}{c}$,

$$\therefore a - b : a :: c - d : c,$$

and $a : a - b :: c : c - d$, *invertendo*.

9. If $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$.

For $\frac{a+b}{b} = \frac{c+d}{d}$, by Prop. 6,

and $\frac{a-b}{b} = \frac{c-d}{d}$, „ 7,

$\therefore \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}$, or $\frac{a+b}{a-b} = \frac{c+d}{c-d}$,

$$\therefore a + b : a - b :: c + d : c - d.$$

10. If $a : b :: c : d :: e : f$, &c., then $a : b :: a + c + e +$ &c. $: b + d + f +$ &c. (Euclid, Book V. Prop. 12.)

$\because \frac{a}{b} = \frac{c}{d}$, $\therefore ad = bc$, $\because \frac{a}{b} = \frac{e}{f}$, $\therefore af = be$, and $ab = ba$.

Hence $ab + ad + af = ba + bc + be$, by addition,

or $a(b + d + f) = b(a + c + e)$,

$\therefore a : b :: a + c + e : b + d + f$, by Prop 2,

and similarly when more quantities are taken.

11. If $a : b :: c : d$, then $ma : mb :: \frac{c}{n} : \frac{d}{n}$,

and $ma : \frac{b}{n} :: mc : \frac{d}{n}$.

For $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{ma}{mb} = \frac{\frac{c}{n}}{\frac{d}{n}}$ (Art. IX.),

$$\therefore ma : mb :: \frac{c}{n} : \frac{d}{n}$$

Again, $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{ma}{b} = \frac{mc}{d}$, $\frac{ma}{\frac{b}{n}} = \frac{mc}{\frac{d}{n}}$,

$$\therefore ma : \frac{b}{n} :: mc : \frac{d}{n}.$$

12. If $\left.\begin{array}{l} a : b :: c : d \\ \text{and } e : f :: g : h \end{array}\right\}$, then $ae : bf :: cg : dh$,

For $\frac{a}{b} = \frac{c}{d}$, $\frac{e}{f} = \frac{g}{h}$, $\therefore \frac{ae}{bf} = \frac{cg}{dh}$, by multiplication,

$$\therefore ae : bf :: cg : dh,$$

and similarly for any number of proportions.

13. If $a : b :: c : d$, then $a^n : b^n :: c^n : d^n$, n being either integral or fractional.

For $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a^n}{b^n} = \frac{c^n}{d^n}$, $\therefore a^n : b^n :: c^n : d^n$.

14. If $a : b :: b : c$, then $a : c :: a^2 : b^2$. (Euclid, Book V Def. 10.)

For $\frac{a}{b} = \frac{b}{c}$, $\therefore \frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$,

$$\therefore a : c :: a^2 : b^2$$

15. If $a : b :: b : c :: c : d$, then $a : d :: a^3 : b^3$. (Euclid, Book V. Def 11.)

$$\text{For } \frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}, \text{ and } \frac{a}{b} = \frac{b}{c} = \frac{c}{d},$$

$$\therefore \frac{a}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}, \quad \therefore a : d :: a^3 : b^3.$$

In this proposition, and the preceding, the given quantities are in *continued proportion.*

VARIATION.

XXI. When two quantities have such a mutual relation that one, being changed the other is changed in the *same proportion*, they are said to *vary directly* as each other.

Suppose A and B to have such a relation to each other, that if the value of A becomes a, B will have such a value b, as that $A : a :: B : b$; then A varies directly as B, or $A \propto B$.

For instance, let Δ represent the area of a triangle, b the base, and p the perpendicular altitude; then if p be constant, $\Delta \propto b$. (Euclid, Book VI. Prop. 1.)

Suppose A and B to have such a relation to each other, that A being changed to a, B is changed to b in such a manner that $A : a :: \frac{1}{B} : \frac{1}{b}$; then A varies inversely as B, or $A \propto \frac{1}{B}$.

Let Δ, b and p represent the same magnitudes as before, then $\Delta = \frac{1}{2} bp$ (Euclid, Book I. Prop. 41.); and if Δ is constant, and b increases, p must decrease in the same proportion: thus if b becomes mb, p will become $\frac{p}{m}$, $\therefore$ *b varies inversely* as p, or $b \propto \frac{1}{p}$.

Prop. If A varies as B, A is equal to B multiplied by some invariable quantity.

For $\because$ A : a :: B : b, $\therefore$ A : a :: mB : mb, $\therefore$ A : mB :: a : mb, alternando; if, therefore, m be so taken that A $= m$B, then, in all cases $a = mb$.

Conversely, if A $= m$B, and m is constant, then A $\propto$ B.

*** This proposition is of very extensive use in Variation, since the variation is convertible into an equation.

Examples in Inequalities, Ratio, Proportion, and Variation.

1. Which is greater, the ratio of 15 : 17 or 17 : 19?

$$\left.\begin{aligned}\frac{15}{17} &= \frac{15}{17} \times \frac{19}{19} = \frac{285}{323} \\ \frac{17}{19} &= \frac{17}{19} \times \frac{17}{17} = \frac{289}{323}\end{aligned}\right\} \quad \therefore 17 : 19 \text{ is greater.}$$

2. Prove that $a^3 + b^3 : a^2 + b^2$ is greater than $a^2 + b^2 : a + b$.

$$\frac{a^3 + b^3}{a^2 + b^2} = \frac{a^3 + b^3}{a^2 + b^2} \times \frac{a + b}{a + b} = \frac{a^4 + a^3b + ab^3 + b^4}{(a^2 + b^2)(a + b)},$$

$$\frac{a^2 + b^2}{a + b} = \frac{a^2 + b^2}{a + b} \times \frac{a^2 + b^2}{a^2 + b^2} = \frac{a^4 + 2a^2b^2 + b^4}{(a^2 + b^2)(a + b)},$$

$$\therefore a^3 + b^3 : a^2 + b^2 > \text{ or } < a^2 + b^2 : a + b$$

according as $a^3b + ab^3 >$ or $< 2a^2b^2$,

or, as $a^2 + b^2 >$ or $< 2ab$,

$\therefore a^3 + b^3 : a^2 + b^2 > a^2 + b^2 : a + b$. (Art. XVIII. *Ex.* 1.)

3. If $x : y$ in the duplicate ratio of $a : b$, and $a : b$ in the sub-duplicate ratio of $a + x : a - y$, then $2x : a :: x - y : y$.

$$x : y :: a^2 : b^2, \ \ldots\ldots\ldots\ldots\ldots (1) \qquad \therefore \frac{x}{y} = \frac{a^2}{b^2},$$

$$\sqrt{(a + x)} : \sqrt{(a - y)} :: a : b, \ (2) \qquad \therefore \frac{a + x}{a - y} = \frac{a^2}{b^2},$$

$$\therefore \frac{a+x}{a-y} = \frac{x}{y}, \qquad \therefore a + x : a - y :: x : y.$$

$x + y : a - y :: x - y : y$ ... dividendo, (Art. XX.)

$x + y : x - y :: a - y : y$ ... alternando,

$2x : x - y :: a : y$ .... componendo,

$\therefore 2x : a :: x - y : y$ .... alternando.

4. If $x^2 + y^2 : xy :: 13 : 6$, and $x^2 - y^2 = 20$; $x = 6, y = 4$.

By Art. XX., $x^2 + y^2 : 2xy :: 13 : 12$,

$$x^2 + 2xy + y^2 : x^2 - 2xy + y^2 :: 25 : 1,$$

$$x + y : x - y :: 5 : 1,$$

$$5x - 5y = x + y, \qquad \therefore 4x = 6y, \qquad \therefore x = \tfrac{3}{2}y;$$

$$\tfrac{9}{4}y^2 - y^2 = 20, \quad 9y^2 - 4y^2 = 80,$$

$$5y^2 = 80, \qquad \therefore y^2 = 16, \qquad \therefore y = \pm 4,$$

$$x = \tfrac{3}{2}y = \tfrac{3}{2} \times \pm 4 = \pm 6.$$

5. If $y = p + q + r$, where p is constant, q varies as x, and r as $\frac{1}{x}$, and if, when $x = 1, 2, 3$, $y = 3, 5\frac{1}{2}, 7$; show that $y = 5 + x - \frac{3}{x}$.

Let $q = ax$, $r = \frac{b}{x}$, $\qquad \therefore y = p + ax + \frac{b}{x}$.

But if $x = 1$, $y = 3$, $\qquad \therefore 3 = p + a + b$ .... (1)

" $x = 2$, $y = \frac{11}{2}$, $\qquad \frac{11}{2} = p + 2a + \frac{b}{2}$ .... (2)

" $x = 3$, $y = 7$, $\qquad 7 = p + 3a + \frac{b}{3}$ .... (3)

$$\left.\begin{aligned}(2)-(1),\ \frac{5}{2} &= a - \frac{b}{2} \\ (3)-(2),\ \frac{3}{2} &= a - \frac{b}{6},\end{aligned}\right\}\ \text{Subtracting,}$$

$$1 = -\frac{b}{3}, \qquad \therefore b = -3.$$

$$\frac{5}{2} = a + \frac{3}{2}, \qquad \therefore a = 1.$$

$$3 = p + 1 - 3, \quad \therefore p = 5,$$

$$\therefore y = 5 + x - \frac{3}{x}.$$

6. If $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$, show that

xy is greater than $ac + bd$, and than $ad + bc$.

$$x^2y^2 = a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2,$$

$$= a^2c^2 + 2acbd + b^2d^2 + a^2d^2 - 2adbc + b^2c^2,$$

$$= (ac + bd)^2 + (ad - bc)^2,$$

$\therefore xy = \sqrt{\{(ac + bd)^2 + (ad - bc)^2\}}$, which is $> ac + bd$.

Again, $x^2y^2 = a^2d^2 + b^2c^2 + a^2c^2 + b^2d^2$,

$$= a^2d^2 + 2adbc + b^2c^2 + a^2c^2 - 2acbd + b^2d^2,$$

$$= (ad + bc)^2 + (ac - bd)^2,$$

$\therefore xy = \sqrt{\{(ad + bc)^2 + (ac - bd)^2\}}$, which is $> ad + bc$.

7. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ &c., prove that

(1) $\frac{a}{b} = \sqrt{\frac{a^2 + b^2 + c^2 + \text{\&c.}}{b^2 + d^2 + f^2 + \text{\&c.}}}$, (2) $\frac{a}{b} = \frac{ma + nc + pe + \text{\&c.}}{mb + nd + pf + \text{\&c.}}$

$$(1)\ \frac{a}{b}=\frac{c}{d}, \qquad \therefore ad=bc, \qquad \therefore a^2d^2=b^2c^2,$$

$$\frac{a}{b}=\frac{e}{f}, \qquad \therefore af=be, \qquad \therefore a^2f^2=b^2e^2,$$

Hence, by addition, $a^2b^2+a^2d^2+a^2f^2=a^2b^2+b^2c^2+b^2e^2$,

or, $a^2(b^2+d^2+f^2)=b^2(a^2+c^2+e^2)$,

$$\therefore \frac{a^2}{b^2}=\frac{a^2+c^2+e^2}{b^2+d^2+f^2}, \quad \therefore \frac{a}{b}=\sqrt{\frac{a^2+c^2+e^2+\&c.}{b^2+d^2+f^2+\&c.}}$$

$$(2)\quad \frac{a}{b}=\frac{ma}{mb}, \qquad \frac{a}{b}=\frac{nc}{nd}, \qquad \frac{a}{b}=\frac{pe}{pf},$$

$$amb=bma, \qquad and=bnc, \qquad apf=bpe$$

Hence, $amb+and+apf=bma+bnc+bpe$,

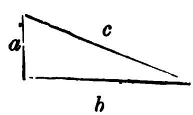
or, $a(mb+nd+pf)=b(ma+nc+pe)$,

$$\therefore \frac{a}{b}=\frac{ma+nc+pe+\&c.}{mb+nd+pf+\&c.}.$$

8. The ratio of the sum of the sides of a right-angled triangle to the hypothenuse is $m:n$; show that m cannot be greater than $n\sqrt{2}$.

Let a and b be the sides, c the hypothenuse, then $a+b:c::m:n$,

$a^2+b^2=c^2$, (Euclid, Book 1. Prop. 47.)



$$\frac{a+b}{c}=\frac{m}{n}, \quad \therefore a+b=c\,.\,\frac{m}{n},$$

$$\left.\begin{array}{l} a^2+2ab+b^2=c^2\,.\,\dfrac{m^2}{n^2} \\ 2a^2 \qquad\quad +2b^2=2c^2 \end{array}\right\} \text{ Subtract.}$$

$$a^2-2ab+b^2=2c^2-c^2\,.\,\frac{m^2}{n^2}=\frac{c^2}{n^2}(2n^2-m^2),$$

$$\left.\begin{aligned} \therefore a - b &= \frac{c}{n}\sqrt{(2n^2 - m^2)} \\ a + b &= \frac{c}{n}.m \end{aligned}\right\} \text{Add and subtract.}$$

$$\therefore 2a = \frac{c}{n}\{m \pm \sqrt{(2n^2 - m^2)}\},$$

$$2b = \frac{c}{n}\{m \mp \sqrt{(2n^2 - m^2)}\}.$$

Now these values of $2a$ and $2b$ become impossible when $m^2 > 2n^2$, or $m > n\sqrt{2}$,

$$\therefore m \text{ cannot be greater than } n\sqrt{2}.$$

Examples.

1. Which is greater, $3 : 7$ or $4 : 9$? Ans. $4 : 9$.

Compound the ratios $2 : 9$ and $12 : 5$. Ans. $8 : 15$.

3. Compound the ratios $m : 6x^2$, $3y^2 : n$, $x^2 : 2y^2$. Ans. $m : 4n$.

4. If $x > y$; which is greater, $x - y$ or $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2$? Ans. $x -$.

5. If $A \propto B$, and $B \propto C$, show that $A \propto C$.

6. If $a : b :: c : d$, prove that
$$ma \pm nb : pa \pm qb :: ma \pm nd : pc \pm qd.$$

7. If $x = y + 12$, and $\sqrt{xy} : 4 :: \frac{1}{2}(x + y) : 5$, show that $x = 16$ and $y = 4$.

8. If $a > b$, show that $\sqrt{(a^2 - b^2)} + \sqrt{\{a^2 - (a - b)^2\}} > a$.

9. $y \propto x$, and when $x = 2$, $y = 10$, show that $y = 5x$.

10. Find two numbers in the ratio of 3 to 2, whose sum multiplied by their product is equal to 12 times the difference of their squares. Ans. 6 and 4.

11 Let $y^2 \propto a^2 - x^2$, and when $x = 0$, suppose $y = b$; then shall $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$.

12. If $x : y :: a^3 : b^3$, and $\sqrt[3]{(c + x)} : \sqrt[3]{(d + y)} :: a : b$, show that $dx = cy$.

13. The sum of three numbers in continued proportion is 52, and the sum of the extremes : the mean :: 10 : 3. Find them. Ans. 4, 12, 36.

CHAPTER VIII.

ARITHMETICAL PROGRESSION.

XXII. When the terms of a series of quantities *increase* or *decrease* by a common difference, the series is called an *Arithmetical Progression*. Thus, 1, 3, 5, 7, 9 is an arithmetical progression whose first term is 1, last term 9, common difference 2, number of terms 5, and sum 25. If this series were written thus, 9, 7, 5, 3, 1, its first term would be 9, last term 1, and common difference -2.

Prop. To find the nth term and the sum of any arithmetical progression.

Let a be the first term, l the last or nth term, d the common difference, n the number of terms, and S the sum of the series: then the series will be

$$a, (a + d), (a + 2d), (a + 3d) \ldots\ldots \{a + (n - 1)d\}.$$

Now any term of this series = the first term + the common difference multiplied by a factor less by unity than the place of the term; and the last term $l = a + (n - 1)d$. The same series written backwards would be

$$l, (l - d), (l - 2d), (l - 3d) \ldots\ldots a.$$

$$\text{Hence } S = a + (a + d) + (a + 2d) \ldots\ldots + (l - 2d) + (l - d) + l,$$

$$\text{and } S = l + (l - d) + (l - 2d) \ldots\ldots + (a + 2d) + (a + d) + a,$$

$$\therefore 2S = (l + a) + (l + a) + (l + a) \ldots\ldots + (l + a) + (l + a) + (l + a)$$

$$= (l + a) \text{ repeated } n \text{ times,}$$

$$\therefore 2S = (l - a) \cdot n, \quad \therefore S = (l + a) \cdot \frac{n}{2}.$$

$$\text{But } \because l = a + (n - 1)d, \quad \therefore l + a = 2a + (n - 1)d,$$

$$\therefore S = \{2a + (n - 1)d\} \cdot \frac{n}{2} = na + \frac{n(n - 1)d}{2}$$

An *arithmetic mean* between two quantities = half their sum; thus if $m = \frac{a+b}{2}$, m is an arithmetic mean between a and b.

When the number of terms of an arithmetic series is *odd*, the middle term = half the sum of any two terms equidistant from it.

Examples.

1. Find the sum of $2 + 5 + 8 +$ &c., to 17 terms.

Here $a = 2$, $d = 3$, $n = 17$,

$$\therefore S = na + \frac{n(n-1)d}{2} = 17 \times 2 + \frac{17 \times 16 \times 3}{2}$$

$$= 34 + 408 = 442. \quad \text{Ans.}$$

2. Find the sum of $\frac{1}{2} + \frac{3}{8} + \frac{1}{4} +$ &c., to 20 terms.

Here $a = \frac{1}{2}$, $d = -\frac{1}{8}$, $n = 20$,

$$\therefore S = na + \frac{n(n-1)d}{2} = 20 \times \tfrac{1}{2} + \frac{20 \times 19 \times -\frac{1}{8}}{2},$$

$$= 10 - \frac{95}{4} = \frac{40 - 95}{4} = -\frac{55}{4}. \quad \text{Ans.}$$

3. The sum of an arithmetic series is 950, the common difference 3, and the number of terms 25. What is the last term?

Here $S = 950$, $d = 3$, $n = 25$,

$$S = na + \frac{n(n-1)d}{2},$$

$$\therefore 950 = 25a + \frac{25 \times 24 \times 3}{2} = 25a + 900,$$

$$\therefore 50 = 25a, \quad \therefore a = 2, \text{ the first term.}$$

Now $l = a + (n-1)d = 2 + 24 \times 3 = 74$. Ans.

4. Insert 5 arithmetic means between 1 and -1.

Here $a = 1$, $l = -1$, and $\because$ there are 5 mean and 2 extreme terms, $n = 5 + 2 = 7$.

$$l = a + (n - 1)d, \therefore d = \frac{l - a}{n - 1} = \frac{-1 - 1}{7 - 1} = -\frac{2}{6} = -\frac{1}{3}$$

$$\left.\begin{array}{rl} \text{Hence } 1 - \frac{1}{3} = \frac{2}{3} = & \text{the first mean.} \\ \frac{2}{3} - \frac{1}{3} = \frac{1}{3} = & \text{,, second ,,} \\ \frac{1}{3} - \frac{1}{3} = 0 = & \text{,, third ,,} \\ 0 - \frac{1}{3} = -\frac{1}{3} = & \text{,, fourth ,,} \\ -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3} = & \text{,, fifth ,,} \end{array}\right\} \text{Ans.}$$

5. A body of soldiers is drawn up in the form of a hollow equilateral wedge, the ranks of which are 3 deep, and the outer rank consisting of n soldiers. Find the number of soldiers.

First, to find the number, supposing the wedge solid.

$$a = 1, \quad l = n, \quad n = n,$$

$$S = (a + l)\frac{n}{2} = (1 + n)\frac{n}{2} = \frac{n^2 + n}{2}.$$

Next, to find the number that might stand in the hollow part.

$$a = 1, \quad n_1 = n - 9 = l,$$

$$S_1 = (a + l)\frac{n_1}{2} = (1 + n - 9) \cdot \frac{n - 9}{2} = \frac{n^2 - 17n - 72}{2}.$$

Now the former number $-$ the latter $=$ the number sought;

$$\therefore S - S_1 = \frac{n^2 + n}{2} - \frac{n^2 - 17n - 72}{2} = \frac{18n - 72}{2} = 9n - 36.$$

If the ranks be r deep, $n_1 = n - 3r$; and the number of soldiers $= \frac{3r}{2}(2n + 1 - 3r)$.

6. Find the sum of the series $1^3 + 2^3 + 3^3 \ldots\ldots + n^3$.

$1^3 + 2^3 \quad + \quad 3^3 \quad + \quad 4^3 \quad \ldots + n^3 =$

$1 + (3 + 5) + (7 + 9 + 11) + (13 + 15 + 17 + 19) \ldots + n^3,$

of which there are n sets, and the nth set contains n terms.

Hence to find the entire number of terms,

$$= 1 \quad d = 1, \quad n = n$$

$$\therefore \{2 + n - 1\} \cdot \frac{n}{2} = \frac{n^2 + n}{2} = \text{the number of terms.}$$

Now, to find the sum of the whole series,

$$a = 1, \quad d = 2, \quad n_1 = \frac{n^2 + n}{2},$$

$$\therefore S = \left\{2 + 2\left(\frac{n^2 + n}{2} - 1\right)\right\} \cdot \frac{n^2 + n}{4} = \left(\frac{n^2 + n}{2}\right)^2.$$

Since $\frac{n^2 + n}{2} = 1 + 2 + 3 \ldots + n$,

$$\therefore 1^3 + 2^3 + 3^3 \ldots + n^3 = (1 + 2 + 3 \ldots + n)^2.$$

7. S_1, S_2, &c., S_p are the sums of p arithmetical progressions, each continued to n terms; the first terms are 1, 2, 3, &c., and the common differences 1, 3, 5, &c.: prove that

$$S_1 + S_2 + \&c. + S_p = \tfrac{1}{2} np\,(np + 1).$$

$$\left.\begin{array}{l} S_1 = 1, 2, 3 \ldots \\ S_2 = 2, 5, 8 \ldots \\ S_3 = 3, 8, 13 \ldots \end{array}\right\} \&c., \text{ to } n \text{ terms}, \therefore \left\{\begin{array}{lll} \text{1st com. dif.} & 1 = 2 \times 1 - 1, \\ \text{2nd} \quad ,, & 3 = 2 \times 2 - 1, \\ \text{3rd} \quad ,, & 5 = 2 \times 3 - 1, \end{array}\right.$$

$$\therefore p\text{th} \quad ,, \qquad = 2 \times p - 1,$$

$\therefore S_p = p,\ 3p - 1,\ 5p - 2$, &c., to n terms $=$ the pth series.

Now

$$S_1 = na + \frac{n(n-1)d}{2} = n + \frac{n(n-1)}{2} \cdot 1 \qquad = \frac{n^2 + n}{2},$$

$$S_2 = \ldots\ldots\ldots\ldots\ldots = 2n + \frac{n(n-1)}{2} \cdot 3 \qquad = \frac{3n^2 + n}{2},$$

$$S_3 = \ldots\ldots\ldots\ldots\ldots = 3n + \frac{n(n-1)}{2} \cdot 5 \qquad = \frac{5n^2 + n}{2},$$

$$\vdots$$

$$S_p = \ldots\ldots\ldots\ldots\ldots = pn + \frac{n(n-1)}{2} \cdot (2p - 1) = \frac{(2p-1)n^2 + n}{2},$$

the com. diff. of which is n^2, $\therefore$ they are in arith. prog., and their sum is the sum sought.

$$a = \frac{n^2 + n}{2}, \quad l = \frac{(2p - 1)n^2 + n}{2}, \quad n = p,$$

$$\therefore s = (a + l)\frac{n}{2} = \left\{\frac{n^2 + n}{2} + \frac{(2p - 1)n^2 + n}{2}\right\} \cdot \frac{p}{2} = \tfrac{1}{2}np(np + 1).$$

8. A party of foot begin their march at 6 in the morning, and travel $3\frac{1}{2}$ miles an hour; 3 hours after a troop of horse follow them from the same place, and travel $3\frac{1}{2}$ miles the first hour, 4 miles the next, $4\frac{1}{2}$ the third, and so on. In what time will they overtake the foot?

Let x be the number of hours in which the horse overtake the foot,

$$\text{then for the horse, } a = \frac{7}{2},\ d = \frac{1}{2},\ n = x,$$

$$s = \frac{n}{2}\{2a + (n - 1)d\} = \frac{x}{2}\left\{7 + (x - 1) \cdot \frac{1}{2}\right\}$$

$$= \frac{x}{2}\left\{7 + \frac{x}{2} - \frac{1}{2}\right\} = \frac{x}{2}\left\{\frac{13}{2} + \frac{x}{2}\right\}$$

$$= \frac{13x + x^2}{4} = \text{miles the horse march.}$$

Again, $x + 3 =$ hours the foot march.

$$\frac{7}{2}(x + 3) = \text{miles the foot march,}$$

$$\therefore \frac{13x + x^2}{4} = \frac{7(x + 3)}{2}, \text{ or } 13x + x^2 = 14x + 42,$$

$$x^2 - x = 42, \quad \therefore x = 7 \text{ hours. Ans.}$$

9. Prove that the latter half of $2n$ terms of an arithmetical series $= \frac{1}{3}$ of the sum of $3n$ terms of the same series.

Let a, $a + d$, $a + 2d$, &c., be the series.

Then, to find the sum of the latter half of $2n$ terms,

$a = a$, $d = d$, $n = n$, $l = a + (n - 1)d$, the nth or last term of the first half.

$\therefore a + (n - 1)d + d$, or $a + nd$ is the first term of the 2nd half, and $a_1 = a + nd$, $d = d$, $n = n$

Hence $s = \frac{n}{2}\{2a_1 + (n-1)d\} = \frac{n}{2}\{2(a+nd) + (n-1)d\} =$

$$\frac{n}{2}\{2a + 2nd + nd - d\} = \frac{n}{2}\{2a + 3nd - d\} =$$

sum of latter half of $2n$ terms.

Now to find the sum of $3n$ terms,

$$a = a, \quad d = d, \quad n_1 = 3n,$$

$$\therefore s_1 = \frac{n_1}{2}\{2a + (n_1 - 1)d\} = \frac{3n}{2}\{2a + (3n-1)d\} =$$

$$\frac{3n}{2}\{2a + 3nd - d\},$$

$$\therefore \frac{n}{2}\{2a + 3nd - d\} \text{ is } \tfrac{1}{3} \text{ of the sum of } 3n \text{ terms.}$$

Hence, the latter half of $2n$ terms $= \frac{1}{3}$ the sum of $3n$ terms.

10. If $x^2 + (x+1)^2 + (x+2)^2 + \ldots$ to 9 terms $= 501$, find x.

Let $z - 4 = x$, then the series becomes

$$(z-4)^2 + (z-3)^2 + (z-2)^2 + (z-1)^2 + z^2 + (z+1)^2 + (z+2)^2$$
$$+ (z+3)^2 + (z+4)^2 = 501,$$

$$2z^2 + 32 + 2z^2 + 18 + 2z^2 + 8 + 2z^2 + 2 + z^2 = 501,$$

$$9z^2 + 60 = 501, \quad 9z^2 = 441, \quad 3z = 21,$$

$$z = 7, \quad \therefore x = 3.$$

11. The sum of 3 numbers in arithmetical progression is 30, and the sum of their squares is 308; find them.

Let $x - y$, x, $x + y$, be the numbers,

then $x - y + x + x + y = 30$, or $3x = 30$, $\therefore x = 10$.

$$x^2 - 2xy + y^2 + x^2 + x^2 + 2xy + y^2 = 308,$$

$$3x^2 + 2y^2 = 308, \text{ or } 2y^2 = 308 - 3x^2 =$$

$$308 - 300 = 8, \quad \therefore y^2 = 4, \quad y = 2 = \text{com. diff.}$$

$$\left.\begin{array}{r} \text{Hence } 10 - 2 = 8. \\ 10. \\ 10 + 2 = 12. \end{array}\right\} \text{Ans.}$$

12. There are 4 numbers in arithmetical progression, whose sum is 24, and product 945; find them.

Let $x - 3y$, $x - y$, $x + y$, $x + 3y$ be the numbers,

then the sum $4x = 24$, $\therefore x = 6$,

the product $(x - 3y)(x + 3y)(x - y)(x + y) = 945$,

or $(x^2 - 9y^2)(x^2 - y^2) = 945$,

or $x^4 - 10x^2y^2 + 9y^4 = 945$,

$\therefore$ by substitution, $1296 - 360y^2 + 9y^4 = 945$.

$9y^4 - 360y^2 = -351$, $\therefore y = 1$.

$$\left.\begin{aligned} \text{Hence } x - 3y &= 6 - 3 = 3. \\ x - y &= 6 - 1 = 5. \\ x + y &= 6 + 1 = 7. \\ x + 3y &= 6 + 3 = 9. \end{aligned}\right\} \text{Ans.}$$

Examples.

1. Find the sum of n terms of $1 + 2 + 3 + 4 +$ &c. Ans. $\frac{1}{2}n(n + 1)$

2. The sum of $9 + 15 + 21$, &c., to 10 terms $= 360$.

3. Find the 20th term, and the sum of 20 terms of $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} +$ &c. Ans $-2\frac{2}{3}$, and $-21\frac{2}{3}$.

4. Sum $1 + 8 + 15 +$ &c., to 100 terms Ans. 34750.

5. Sum $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} +$ &c., to 12 terms. Ans. $-1\frac{1}{2}$.

6. The 9th term of the series, 7, $5\frac{1}{2}$, 4, &c. $= -5$.

7. Required the 24th term, and the sum of 24 terms, of $\frac{7}{12} + \frac{2}{3} + \frac{3}{4} +$ &c. Ans. $2\frac{1}{2}$, and 37.

8. Given $a = 3$, $l = 17$, $n = 29$; find the series. Ans. 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, &c.

9. The 100th term of the series 1, 9, 17, &c. $= 793$.

10. The sum of $\frac{1}{2} + \frac{3}{4} + 1 +$ &c., to 10 terms $= 16\frac{1}{4}$.

11. Find the sum of the series 198, 193, 188, &c, to 40 terms Ans 4020.

12. The 20th term of the series $\frac{1}{2}$, $\frac{1}{6}$, $-\frac{1}{6}$, &c. $= -5\frac{5}{6}$.

13. The sum of $\frac{1}{2} - \frac{2}{3} - \frac{11}{6} -$ &c., to 13 terms $= -84\frac{1}{2}$.

14. Sum $\frac{5}{7} + 1 + 1\frac{2}{7} +$ &c., to n terms. Ans. $\frac{n}{7}(n+4)$.

15 Find three arithmetic means between 19 and 35; and five arithmetic means between $2\frac{2}{3}$ and $\frac{2}{3}$.
Ans. 23, 27, 31, and $2\frac{1}{3}$, 2, $1\frac{2}{3}$, $1\frac{1}{3}$, 1.

16. The sum of 15 terms of an arithmetic series is 600, and the common difference is 5; find the first term. Ans. 5.

17. Given $S = 40$, $a = 7$, $d = 2$; find n.
Ans. $n = 4$, or -10.

18. The first term of an arithmetical series is 3, and the sum of 10 terms is 165; find the progression.
Ans. 3, 6, 9, 12, &c.

19. Insert four arithmetic means between $2\frac{1}{2}$ and $6\frac{1}{4}$.
Ans. $3\frac{1}{4}$, 4, $4\frac{3}{4}$, $5\frac{1}{2}$.

20. The sum of 9 terms of an arithmetic progression is 0, and the last term is $-\frac{1}{3}$; find the series.
Ans. $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} +$ &c.

21. Insert three arithmetic means between -1 and 15.
Ans. 3, 7, and 11.

22. The sum of an arithmetic series is $6\frac{7}{8}$, the first term $1\frac{1}{4}$, and the common difference $-\frac{1}{8}$; find the number of terms. Ans. 10.

23. The third and sixth terms of an arithmetical progression are 7 and 16 respectively; find the series.
Ans. 1, 4, 7, 10, &c.

24. There are x arithmetic means between 1 and 31, and the 7th is $\frac{5}{9}$ of the $(x-1)$th mean; find the number of means. Ans. 14.

25. There are three numbers in arithmetic progression, whose sum is 10, and the product of the second and third is $33\frac{1}{3}$; find them. Ans. $-3\frac{1}{3}$, $3\frac{1}{3}$, 10.

26. The sum of three numbers in arithmetical progression is 15, and the sum of the squares 93; find them.

Ans. 2, 5, 8.

27. Find the nth term, and the sum of n terms of

$$\frac{x^2-1}{x} + x + \frac{x^2+1}{x} + \&c.$$

Ans. $x + (n-2)\cdot\frac{1}{x}$, and $nx + \frac{1}{2}n\cdot(n-3)\cdot\frac{1}{x}$.

28. The sum of three numbers in arithmetic progression is 24, and their product 480; find them. Ans. 6, 8, 10.

29. The first term is $n^2 - n + 1$, the common difference 2; find the sum of n terms. Ans. n^3

30. Find four numbers in arithmetical progression, such that the product of the extremes shall be 27, and the product of the means 35. Ans. 3, 5, 7, 9.

31. If the nth and mth terms of an arithmetical progression be m and n respectively, find the number of terms whose sum is $\frac{1}{2}\cdot(m+n)\cdot(m+n-1)$, and the last term of the series. Ans. $m+n$, or $m+n-1$; and 0, or 1

32. If a steam-engine is observed to pass over 4 feet in the first second, and 88 feet in the sixtieth second of its motion, how far will it travel in the first minute, supposing its motion to be increased each second by a constant quantity?

Ans. 2760 feet.

33. Referring to the last example, find the uniform rate of increase, and the time occupied in travelling the first mile.

Ans. $1\frac{25}{59}$ feet; and $83\frac{21}{23}$ seconds, nearly

34. The number of terms of an arithmetic progression is equal to $\frac{1}{2}$ the common difference, the last term is equal to 4 times the first, and the sum of the series is equal to $\frac{3}{4}$ the square of the first term; find the series

Ans. 20, 32, 44, 56, 68, 80

G 2

35 A starts from a certain place, and travels a miles the first day, $2a$ the second, $3a$ the third, &c.; after 4 days, B starts to overtake him, travelling $9a$ miles per day; after how many days will he come up with him? Ans. 4.

36. A number of persons bought a field for 345*l.*, the youngest paying a certain sum, the next 5*l.* more, and so on, in arithmetic progression. The younger half took a portion of the field proportional to the sum they had subscribed, and this they agreed to divide equally, by equalizing their contribution to 22*l.* each; how many persons were there in all? Ans. 10.

37. Find the nth term of an arithmetical progression when the sum of $n+1$ terms is $(n+1)(n+1\frac{1}{3})$. Ans. $2(n-\frac{1}{3})$.

38. The sum of n terms of an arithmetic progression is $pn+qn^2$; find the mth term. Ans. $p+(2m-1)q$.

39. Divide $\frac{n}{7}(n+4)$ into n parts, such that each part shall exceed the one immediately preceding by a fixed quantity. Ans. $\frac{5}{7}$, 1, $1\frac{2}{7}$, $1\frac{4}{7}$, &c.

40 Determine the relation between a, b, and c, that they may be the pth, qth, and rth terms of an arithmetic progression. Ans. $(q-r)a+(r-p)b+(p-q)c=0$.

41. Whereabouts in a coal shaft will the corves meet, if the radius of the roll be $3\frac{1}{2}$ feet, the thickness of the rope $\frac{1}{8}$ foot, and the depth of the pit 1020 feet? Ans. 70·1273 fathoms.

GEOMETRICAL PROGRESSION.

XXIII. When the terms of a series of quantities increase or decrease in a *common ratio*, the series is a *geometric progression;* thus 1, 3, 9, 27, 81 is a geometric progression whose common ratio is 3; also $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$, is a geometrical progression whose common ratio is $\frac{1}{2}$. The common ratio may always be found by dividing any term by the one immediately preceding it.

Prop. To find the nth term and the sum of any geometric progression.

Let a be the first term, l the nth, r the common ratio, and S the sum of the series; then the series will be

$$a,\ ar,\ ar^2,\ ar^3 \ldots\ldots ar^{n-2},\ ar^{n-1},$$

and it is evident that the nth term $l = ar^{n-1}$.

$$\text{Now } S = a + ar + ar^2 \ldots\ldots + ar^{n-2} + ar^{n-1},$$

$$\therefore Sr = \quad ar + ar^2 \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n,$$

$$\therefore Sr - S = ar^n - a, \text{ by subtraction,}$$

$$\text{or } S(r-1) = a(r^n - 1),$$

$$\therefore S = a \cdot \frac{r^n - 1}{r - 1}.$$

Cor. 1. $\because S = \dfrac{ar^n - a}{r - 1}$, and $rl = ar^n$, $\therefore S = \dfrac{rl - a}{r - 1}$.

Cor. 2. $\because S = \dfrac{ar^n}{r - 1} - \dfrac{a}{r - 1}$; if r be a proper fraction, as n increases, r^n will decrease, and when n is increased without limit, r^n will be less than any assignable magnitude, $\therefore \dfrac{ar^n}{r-1}$ may be rejected, and $-\dfrac{a}{r-1}$ or $\dfrac{a}{1-r}$ will express the *limit* of the series.

Hence, when r is a proper fraction and the series is continued ad infinitum, $S = \dfrac{a}{1-r}$.

A *geometric mean* between two quantities = the square root of their product: thus, if a, m, b are in geometric progression, $\dfrac{m}{a} = \dfrac{b}{m}$, $\therefore m^2 = ab$, $\therefore m = \sqrt{ab}$.

Examples.

1. Find the sum of 12 terms of the series 1, 2, 4, 8, &c
Here $a = 1$, $r = 2$, $n = 12$,

$$\therefore S = a \cdot \frac{r^n - 1}{r - 1} = 1 \cdot \frac{2^{12} - 1}{2 - 1} = 4096 - 1 = 4095.$$

2. Sum 6561, 2187, 729, &c., to 6 terms.

Here $a = 6561$, $r = \frac{1}{3}$, $n = 6$,

$$\therefore S = a \cdot \frac{r^n - 1}{r - 1} = a \cdot \frac{\left(\frac{1}{3}\right)^6 - 1}{\frac{1}{3} - 1} = a \cdot \frac{\frac{1}{729} - 1}{\frac{1}{3} - 1}$$

$$= a \cdot \frac{1 - 729}{243 - 729} = a \cdot \frac{-728}{-486} = 6561 \times \frac{364}{243}$$

$$= 27 \times 364 = 9828.$$

3. Sum $\frac{2}{3} - \frac{1}{3} + \frac{1}{6} -$ &c., to infinity.

Here $a = \frac{2}{3}$, $r = -\frac{1}{2}$,

$$\therefore S = \frac{a}{1 - r} = \frac{\frac{2}{3}}{1 + \frac{1}{2}} = \frac{4}{6 + 3} = \frac{4}{9}.$$

4. Insert 4 geometric means between $\frac{1}{3}$ and 81.

Here there are 6 terms, namely, 2 extremes and 4 means, $a = \frac{1}{3}$, $l = 81$, $n = 6$.

Now $l = ar^{n-1}$, $\therefore 81 = \frac{1}{3} r^5$, $r^5 = 243 = 3^5$, $\therefore r = 3$

Hence $\frac{1}{3} \times 3 = 1$, $1 \times 3 = 3$, $3 \times 3 = 9$, $9 \times 3 = 27$, and the means are 1, 3, 9, 27.

5. If an arithmetic mean between a and b be twice as great as a geometric mean, $\frac{a}{b} = \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$.

$\frac{a+b}{2}$ = the arith. mean, $\sqrt{ab}$ = the geom. mean

Now $\frac{a+b}{2} = 2\sqrt{ab}$, $\therefore a + b = 4\sqrt{a} \cdot \sqrt{b}$,

$$\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}} = 4, \quad \therefore \frac{a}{b} + 2 + \frac{b}{a} = 16,$$

$$\frac{a}{b} - 2 + \frac{b}{a} = 12,$$

$$\left.\begin{array}{l} \frac{\sqrt{a}}{\sqrt{b}} - \frac{\sqrt{b}}{\sqrt{a}} = 2\sqrt{3}, \\ \frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}} = 4, \end{array}\right\} \therefore \begin{array}{l} 2 \cdot \frac{\sqrt{a}}{\sqrt{b}} = 4 + 2\sqrt{3}. \\ 2 \cdot \frac{\sqrt{b}}{\sqrt{a}} = 4 - 2\sqrt{3}. \end{array}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = 2 + \sqrt{3}, \quad \frac{\sqrt{b}}{\sqrt{a}} = 2 - \sqrt{3}.$$

Hence $\frac{\sqrt{a}}{\sqrt{b}} \div \frac{\sqrt{b}}{\sqrt{a}} = \frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$.

6. The sum of a series to infinity is 2, and the sum of the squares of the terms of the same series is $\frac{4}{3}$; find a and r.

$$a + ar + ar^2 + ar^3 + \&c. = \frac{a}{1-r} = 2, \quad (1).$$

$$a^2 + a^2r^2 + a^2r^4 + a^2r^6 + \&c. = \frac{a^2}{1-r^2} = \frac{4}{3}, \quad (2).$$

Dividing (2) by (1), $\frac{a}{1+r} = \frac{2}{3}$, $\therefore a = \frac{2}{3}(1+r)$.

Also (1) $a = 2(1-r)$; and, equating these values of a,

$$\frac{2}{3}(1+r) = 2(1-r), \quad 1 + r = 3 - 3r,$$

$$4r = 2. \quad \therefore r = \frac{1}{2}, \text{ and } a = 2 - 2r = 2 - 1 = 1.$$

7. If $S = 1 + R + R^2 + R^3 +$ &c., to infinity, and
$s = 1 + r + r^2 + r^3 +$ &c., to infinity, prove that the sum of $1 + Rr + R^2r^2 +$ &c., to infinity $= \dfrac{Ss}{S + s - 1}$.

Now, $1 + Rr + R^2r^2 + \text{\&c.} = \dfrac{1}{1 - Rr}$

$$S = \frac{1}{1 - R},\ s = \frac{1}{1 - r},\ \therefore S - SR = 1,\ s - sr = 1,$$

$$SR = S - 1,\ sr = s - 1,\ \therefore R = 1 - \frac{1}{S},\ r = 1 - \frac{1}{s}.$$

$$\therefore Rr = 1 - \frac{1}{S} - \frac{1}{s} + \frac{1}{Ss},\ \therefore 1 - Rr = \frac{1}{S} + \frac{1}{s} - \frac{1}{Ss}.$$

Hence $1 + Rr + R^2r^2 + \text{\&c.} = \dfrac{1}{\dfrac{1}{S} + \dfrac{1}{s} - \dfrac{1}{Ss}} = \dfrac{Ss}{s + S - 1}$.

8. Show that $4 \cdot 5212121$ &c. $= 4\dfrac{86}{165}$.

$$4\ 5212121 \text{ \&c.} = 4 + \frac{5}{10} + \frac{21}{1000} + \frac{21}{100000} + \frac{21}{10000000} + \text{\&c.}$$

But $\dfrac{21}{1000} + \dfrac{21}{100000} + \dfrac{21}{10000000} + \text{\&c.}$

$$= \frac{\dfrac{21}{1000}}{1 - \dfrac{1}{100}} = \frac{21}{1000 - 10} = \frac{21}{990} = \frac{7}{330},$$

$$\therefore 4 \cdot 5212121 \text{ \&c.} = \frac{45}{10} + \frac{7}{330} = \frac{1492}{330} = 4\frac{86}{165}.$$

9. Find the sum of $a - (a + d)x + (a + 2d)x^2 - (a + 3d)x^3 +$ &c., to infinity.

$$S = a - ax - dx + ax^2 + 2dx^2 - ax^3 - 3dx^3 + \text{\&c.},$$
$$Sx = ax - ax^2 - dx^2 + ax^3 + 2dx^3 - ax^4 - 3dx^4 + \text{\&c.},$$
$$\therefore S + Sx = a - dx + dx^2 - dx^3 + dx^4 - dx^5 + \text{\&c.},$$

$$\text{or } (1+x)S = a - d(x - x^2 + x^3 - \&c.) = a - d \cdot \frac{x}{1+x},$$

$$\therefore S = \frac{a}{1+x} - \frac{dx}{(1+x)^2} = \frac{a + (a-d)x}{(1+x)^2}.$$

10. If P be the product, S the sum, and S_1 the sum of the reciprocals of n quantities in geometrical progression; prove that $P^2 = \left(\frac{S}{S_1}\right)^n$.

1st series, $S = a + ar + ar^2 + \dots ar^{n-1} \quad = a \cdot \frac{r^n - 1}{r - 1}$.

2nd series,

$$S_1 = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots \frac{1}{ar^{n-1}} = \frac{1}{a} \cdot \frac{\left(\frac{1}{r}\right)^n - 1}{\frac{1}{r} - 1}$$

$$= \frac{1}{a} \cdot \frac{1 - r^n}{r^{n-1} - r^n} = \frac{1}{ar^{n-1}} \cdot \frac{1 - r^n}{1 - r} = \frac{1}{ar^{n-1}} \cdot \frac{r^n - 1}{r - 1},$$

$$\therefore \frac{S}{S_1} = a^2 r^{n-1}, \qquad \left(\frac{S}{S_1}\right)^n = a^{2n} r^{n(n-1)}.$$

and $P = a \cdot ar \cdot ar^2 \dots \cdot ar^{n-1} = a^n \cdot r^{1+2+3+\dots(n-1)}$

Summing the arithmetic series $1 + 2 + 3 + \dots (n-1)$,

$$S' = \frac{n'}{2}\{2a + (n'-1)d\} = \frac{n-1}{2}\{2 + (n-2)\} =$$

$$\frac{2n - 2 + n^2 - 3n + 2}{2}$$

$$= \frac{n^2 - n}{2} = \frac{n(n-1)}{2},$$

$$\therefore P = a^n \cdot r^{\frac{n(n-1)}{2}}, \qquad P^2 = a^{2n}\ r^{n(n-1)}.$$

$$\text{Hence, } P^2 = \left(\frac{S}{S_1}\right)^n.$$

11. The difference between two numbers = 12; and the arithmetic : to the geometric mean :: 5 : 4; find the numbers.

Let x and y be the numbers, then $x - y = 12$,

$$\frac{x+y}{2} = \text{arithmetic mean}, \quad \sqrt{xy} = \text{geometric mean},$$

$$\frac{x+y}{2} : \sqrt{xy} :: 5 \quad 4, \qquad x + y : 2\sqrt{xy} :: 5 : 4,$$

$$x + 2\sqrt{xy} + y : x - 2\sqrt{xy} + y :: 9 : 1, \text{ Art. XX.}$$

$$\sqrt{x} + \sqrt{y} : \sqrt{x} - \sqrt{y} :: 3 : 1,$$

$$3\sqrt{x} - 3\sqrt{y} = \sqrt{x} + \sqrt{y}, \qquad 2\sqrt{x} = 4\sqrt{y},$$

$$\sqrt{x} = 2\sqrt{y}, \quad x = 4y, \quad 4y - y = 12, \quad 3y = 12.$$

$$\text{Ans. } y = 4, \qquad x = 16.$$

12. If $S = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \text{\&c., to infinity}$

and $S_1 = 1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \text{\&c., to infinity,}$

then $S : S_1 :: 27 : 1.$

$$\left.\begin{aligned} S &= 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} + \text{\&c.} \\ \frac{S}{2} &= \quad\ \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \text{\&c.} \end{aligned}\right\} \text{ Subtract.}$$

$$\therefore \frac{s}{2} = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 + \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 2 + 1 = 3,$$

$$\therefore s = 6.$$

$$\left.\begin{aligned} s_1 &= 1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \frac{9}{16} - \&c. \\ \frac{s_1}{2} &= \quad\;\; \frac{1}{2} - \frac{3}{4} + \frac{5}{8} - \frac{7}{16} + \&c. \end{aligned}\right\} \text{Add}$$

$$\therefore \frac{3s_1}{2} = 1 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \&c$$

$$= \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \&c.\right) - \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \&c.\right)$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{4}} - \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3};$$

$$\therefore s_1 = \frac{2}{9}, \quad \therefore s : s_1 :: 6 : \frac{2}{9}, \quad :: 3 : \frac{1}{9}, \quad :: 27 : 1.$$

13. From a vessel containing 10 gallons of brandy, 1 gallon was drawn out, and a gallon of water poured into the vessel; a gallon of the mixture was then drawn out, and another gallon of water poured in. Now the like process being repeated 10 times, it is required to find how much brandy remained in the vessel, supposing the two fluids were thoroughly mixed each time?

There are 10 gallons of brandy at first;

$\frac{1}{10}$ is drawn;

there are then 9 gallons of brandy,

and 1 gallon of water poured in;

$\frac{9}{10}$ gallon of brandy is drawn,

$$9 - \frac{9}{10} = \frac{81}{10} \text{ gallons of brandy left in,}$$

$$\frac{19}{10} \text{ gallons water left in;}$$

$$\frac{81}{100} \text{ gallons brandy drawn.}$$

Hence $1, \frac{9}{10}, \frac{81}{100}$, &c., to 10 terms = quantity of brandy drawn

$$\text{Here } a = 1; \ r = \frac{9}{10}, \ n = 10,$$

$$\therefore S = \frac{a(r^n - 1)}{r - 1} = \frac{\left(\frac{9}{10}\right)^{10} - 1}{\frac{9}{10} - 1},$$

$$= \frac{\cdot 9^{10} - 1}{\cdot 9 - 1} = \frac{1 - \cdot 3486784401}{1 - \cdot 9},$$

$$= \frac{\cdot 6513215599}{\cdot 1} = 6\cdot 513215599 \text{ gallons drawn.}$$

Subtracting 6·513215599 from 10, we have

3·486784401 gallons. Ans.

14. The sum of three numbers in geometrical progression is 7, and the sum of their reciprocals is $\frac{7}{4}$; find them.

Let $\frac{x}{y}$ be the first term, y the common ratio,

$$\text{then } \frac{x}{y} + x + xy = 7, \qquad \text{and } \frac{y}{x} + \frac{1}{x} + \frac{1}{xy} = \frac{7}{4},$$

$$\left.\begin{array}{l} \frac{1}{y} + 1 + y = \frac{7}{x} \\ y + 1 + \frac{1}{y} = \frac{7}{4}x \end{array}\right\} \quad \begin{array}{l} \therefore \frac{7}{4}x = \frac{7}{x}, \\ x^2 = 4, \qquad \therefore x = 2 \end{array}$$

$$\therefore y + 1 + \frac{1}{y} = \frac{7}{2}, \qquad y - \frac{5}{2} = -\frac{1}{y},$$

$$y^2 - \frac{5}{2}y = -1, \qquad y^2 - \frac{5}{2}y + \left(\frac{5}{4}\right)^2 = \frac{25}{16} - 1 = \frac{9}{16},$$

$$y - \frac{5}{4} = \pm\frac{3}{4}, \qquad \therefore y = \frac{5 \pm 3}{4} = 2;$$

$\therefore$ 1, 2, 4, are the numbers.

15. The sum of four numbers in geometrical progression is 40, and the sum of their squares 820; find them.

Let x, xy, xy^2, xy^3 be the numbers,

then $x + xy + xy^2 + xy^3 = 40$, or $x(1 + y + y^2 + y^3) = 40$,
$x^2 + x^2y^2 + x^2y^4 + x^2y^6 = 820$, or $x^2(1 + y^2 + y^4 + y^6) = 820$,
or $x\{1 + y^2 + y(1 + y^2)\} = 40$, $x(1 + y)(1 + y^2) = 40$, (1),
$x^2\{1 + y^2 + y^4(1 + y^2)\} = 820$, $x^2(1 + y^2)(1 + y^4) = 820$, (2).

Squaring (1) and dividing by (2), we have

$$\frac{x^2(1 + y)^2(1 + y^2)^2}{x^2(1 + y^2)(1 + y^4)} = \frac{1600}{820}, \quad \text{or} \quad \frac{(1 + y)^2 . (1 + y^2)}{1 + y^4} = \frac{80}{41}$$

This equation reduced gives

$$y^2 + \frac{1}{y^2} - \frac{82}{39}\left(y + \frac{1}{y}\right) = \frac{82}{39},$$

whence $y = 3$, $\therefore x = 1$, $\therefore$ 1, 3, 9, 27 are the numbers.

Examples.

1. The sum of $3 + 6 + 12 +$ &c. to 6 terms $= 189$.
2. The sum of $5 + 20 + 80 +$ &c. to 5 terms $= 1705$.
3. The sum of $\frac{3}{2} + 1 + \frac{2}{3} +$ &c. to 6 terms $= 4\frac{17}{162}$.
4. The 8th term of the series $9, -6, 4,$ &c. $= -\frac{128}{243}$.
5. The 6th term of the series $3, \frac{1}{2}, \frac{1}{12}$, &c. $= \frac{1}{2592}$.

6. The sum of $1 - 2 + 2^2 -$ &c. to 10 terms $= - 341$.

7. Determine the 5th term and the sum of 5 terms of $21 - 3 + \frac{3}{7} -$ &c. Ans. $\frac{3}{343}$, and $18\frac{129}{343}$.

8. The 5th term of the series $-\frac{1}{2}, \frac{1}{3}, -\frac{2}{9},$ &c. $= -\frac{8}{81}$.

9. Find expressions for the nth term and the sum of n terms of $\frac{4}{5} + 2 + 5 +$ &c. Ans. $\frac{5^{n-2}}{2^{n-3}}$, and $\frac{1}{1}\left(\frac{5^n - 2^n}{2^{n-3}}\right)$

10. Sum $\frac{3}{5} + \frac{1}{2} + \frac{5}{12} +$ &c. to 8 terms. Ans. $\frac{1288991}{466560}$

11. Find the limit of the series $\frac{1}{4} - \frac{1}{16} + \frac{1}{64} -$ &c $\frac{1}{5}$.

12. Find the limit of the series $1 - \frac{1}{2} + \frac{1}{4} -$ &c. $\frac{2}{3}$.

13. Find the limit of the series $2 - 1\frac{1}{3} + \frac{8}{9} -$ &c. $1\frac{1}{5}$.

14. Find the limit of the series $-3\frac{1}{5} + 1\frac{3}{5} - \frac{4}{5} +$ &c.

Ans. $-2\frac{2}{15}$.

15. Insert three geometric means between $\frac{1}{2}$ and 128.

Ans. ± 2, 8, ± 32.

16. Find two geom. means between 5 and 1080, and three geom. means between 9 and $\frac{1}{9}$. Ans. 30, 180; and 3, 1, $\frac{1}{3}$

17. The sum of an infinite geometric series is 2, and the sum of its first two terms is $1\frac{1}{2}$; find the series.

Ans. $3 - \frac{3}{2} + \frac{3}{4} -$ &c

18. The product of three numbers in geometrical progression is 64, and the sum of the cubes is 584; find them.

Ans. 2, 4, 8.

19. If the sum of a geometrical series continued ad infinitum be twice as great as the sum of n terms, prove that the common ratio is $\left(\frac{1}{2}\right)^{\frac{1}{n}}$.

20. The sum of the arithmetic and geometric means of two numbers is $13\frac{1}{2}$, and if the geometric be subtracted from the arithmetic mean, the remainder will be $1\frac{1}{2}$; find the numbers.
Ans. 3 and 12.

21. A leveret is 100 yards before a greyhound; they start in the same direction, and the greyhound runs 100 times as fast as the leveret; when will the greyhound overtake the leveret?
Ans. When the hound has run $101\frac{1}{99}$ yards.

22. The population of a country increases annually in geometric progression, and in four years was raised from 10,000 to 14,641 souls; by what part of itself was it annually increased?
Ans. $\frac{1}{10}$.

23. In a geometric progression, if the $(p+q)$th term $= m$ and the $(p-q)$th term $= n$, show that the pth term $= \sqrt{mn}$, and the qth term $= m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$. Also, if P be the pth term and Q the qth term, show that the nth term $= \left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$.

24. Determine m and n in terms of a and b, so that $\frac{ma+nb}{m+n}$ may be the arithmetic mean between m and n, and the geometric mean between a and b.

25. Given S the sum, and s^2 the sum of the squares, of the terms of an infinite geometrical progression; show that its sum to n terms $= S\left\{1-\left(\frac{S^2-s^2}{S^2+s^2}\right)^n\right\}$.

HARMONICAL PROGRESSION.

XXIV. When the first of three quantities is to the third as the difference of the first and second is to the difference of the second and third, the three quantities are in *harmonical proportion.*

If $a : c :: a - b : b - c$, and $b : d :: b - c : c - d$,
and $c : e :: c - d : d - e$, &c.,
then a, b, c, d, e, &c., are in *harmonical progression*.

Prop. Let a, b, c, d, &c., be in *harmonical* progression, then their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$. &c., will be in *arithmetical* progression.

For $a : c :: a - b : b - c$, $\therefore ab - ac = ac - bc$,

$$\therefore \frac{ab}{abc} - \frac{ac}{abc} = \frac{ac}{abc} - \frac{bc}{abc}, \text{ or } \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

Similarly, $\frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b}$. Whence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ have a common difference, .. they are in arithmetical progression.

Cor. $\because ab - ac = ac - bc$, $ab + bc = 2ac$, $\therefore b = \frac{2ac}{a+c}$; that is, *an harmonic* mean between two quantities = twice their product divided by their sum.

Examples.

1. Insert two harmonic means between 2 and $\frac{1}{5}$,

$\frac{2}{1}, \frac{1}{5}$ are the extremes of an harmonic series of 4 terms.

$\therefore \frac{1}{2}, \frac{5}{1}$ „ arithmetic „

By Art. XXII., Ex. 4, $d = \frac{l-a}{n-1} = \frac{5 - \frac{1}{2}}{4-1} = \frac{4\frac{1}{2}}{3} = \frac{9}{6} = \frac{3}{2}$,

$$\left.\begin{array}{l} \therefore \frac{1}{2} + \frac{3}{2} = 2 \\ 2 + \frac{3}{2} = \frac{7}{2} \end{array}\right\} \text{the arithmetical means,} \quad \therefore \frac{1}{2}, \frac{2}{7} \text{ are the harmonic means.}$$

2. The sum of three terms of an harmonic series is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$; find the series, and continue it both ways.

Let $x = 2\text{nd}$, $y = 3\text{rd}$,

$$\text{then } \frac{1}{2} + x + y = \frac{13}{12},$$

$$\therefore x + y = \frac{13}{12} - \frac{6}{12} = \frac{7}{12} \text{ .... } (1)$$

But $b = \dfrac{2ac}{a + c}$, by harmonic proportion,

$$\text{i. e., } x = \frac{2 \cdot \frac{1}{2}y}{\frac{1}{2} + y} = \frac{y}{y + \frac{1}{2}} = \frac{2y}{2y + 1},$$

But (1) $\quad x = \dfrac{7}{12} - y, \qquad \therefore \dfrac{2y}{2y + 1} = \dfrac{7}{12} - y.$

Whence $y = \dfrac{1}{4}$, $\qquad \therefore x = \dfrac{1}{3}$,

$$\therefore \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, = \text{\&c. harmonic series};$$

$$\therefore 2, 3, 4, \text{\&c.} = \text{arithmetic series};$$

$$\therefore -1, 0, 1, 2, 3, 4, \text{\&c.}, = \text{arithmetic series continued};$$

$$\therefore \frac{1}{-1}, \frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \text{\&c.}, = \text{harmonic series continued};$$

$$\therefore -1, \infty, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \text{\&c.} = \text{Ans}$$

3. The sum of three terms of an harmonic series is 11, and the sum of their squares is 49; find the numbers.

Let x, y be the extremes, then $\dfrac{2xy}{x + y}$ is the mean;

$$x + y + \frac{2xy}{x + y} = 11, \text{ .... } (1)$$

$$x^2 + y^2 + \frac{4x^2y^2}{(x + y)^2} = 49, \text{ .... } (2)$$

$$(1)\ \frac{2xy}{x+y} = 11 - (x+y),$$

$$\left.\begin{array}{l} \therefore\ \dfrac{4x^2y^2}{(x+y)^2} = 121 - 22(x+y) + x^2 + 2xy + y^2 \\ (2)\ \dfrac{4x^2y^2}{(x+y)^2} = 49 \qquad\qquad\qquad\quad - x^2 \qquad\ - y^2 \end{array}\right\}\text{Subtract.}$$

$$0 = 72 - 22(x+y) + 2x^2 + 2xy + 2y^2,$$

$$\left.\begin{array}{l} \therefore\ x^2 + xy + y^2 = 11(x+y) - 36 \\ (1)\ x^2 + 4xy + y^2 = 11(x+y) \end{array}\right\}\text{Subtract.}$$

$$3xy \quad = \quad 36, \ \therefore xy = 12,$$

$$\left.\begin{array}{l} x^2 + xy + y^2 - 11(x+y) = -36 \\ xy \qquad\qquad\qquad\qquad\quad = 12 \end{array}\right\}$$

$$x^2 + 2xy + y^2 - 11(x+y) = -24, \ \therefore x + y = 8.$$

Whence $x = 6$, $\therefore y = 2$, $\therefore$ 2, 3, 6 are the numbers.

Examples.

1. Insert two harmonic means between 2 and 4. Ans. $2\frac{2}{5}$, 3.

2. Insert three harmonic means between 4 and $1\frac{5}{7}$. Ans. 3, $2\frac{2}{5}$, and 2.

3. Find the arithmetic, geometric, and harmonic means between $3\frac{3}{8}$ and $1\frac{1}{2}$. Ans. $2\frac{7}{16}$, $2\frac{1}{4}$, $2\frac{1}{13}$.

4. Insert two harmonic means between $\frac{1}{5}$ and $\frac{1}{11}$. Ans. $\frac{1}{7}$ and $\frac{1}{9}$.

5. Continue to three terms each way the series $1\frac{1}{2}$, $2\frac{1}{7}$, $3\frac{3}{4}$. Ans. $\frac{15}{19}$, $\frac{15}{16}$, $1\frac{2}{13}$, ... 15, $-7\frac{1}{2}$, -3.

6. Find two numbers whose difference is 8, and the harmonic mean between them $1\frac{4}{5}$. Ans. 9 and 1, or $\frac{4}{5}$ and $-7\frac{1}{5}$.

7. An harmonic mean between two numbers is 2, and the sum of the extremes $= 4\frac{4}{35}$; find the series. Ans. $2\frac{2}{5}$, 2, $1\frac{5}{7}$.

8. The sum of the arithmetic and harmonic means of two numbers is $12\frac{5}{7}$, and if the harmonic be subtracted from the arithmetic mean, the remainder will be $1\frac{2}{7}$; find the numbers. Ans. 4 and 10

9. If the geometric mean between x and y : the harmonic mean :: $m : n$, then $x : y :: m + \sqrt{(m^2 - n^2)} : m - \sqrt{(m^2 - n^2)}$.

10. If the geometric mean between a and b be to the harmonic as $4 : 3$; then

$$a : b :: 4 + \sqrt{7} : 4 - \sqrt{7}.$$

CHAPTER IX.

BINOMIAL SURDS.

Propositions.

XXV. 1. The square root of a quantity cannot be partly rational, and partly a quadratic surd.

If possible, let $\sqrt{x} = a + \sqrt{y}$, then $x = a^2 + 2a\sqrt{y} + y$,

$\therefore x - y - a^2 = 2a\sqrt{y}$, $\quad \therefore \dfrac{x - y - a^2}{2a} = \sqrt{y}$; or, a rational quantity = an irrational quantity, which is impossible.

2. If $a + \sqrt{x} = b + \sqrt{y}$ be an equation between rational quantities and quadratic surds, then $a = b$, and $\sqrt{x} = \sqrt{y}$.

For $\sqrt{x} = b - a + \sqrt{y}$, and if $b - a$ be not $= 0$, $\sqrt{x}$ would be partly rational and partly a quadratic surd, which is impossible, by Proposition 1, $\quad \therefore b - a = 0$, or $b = a$, $\therefore \sqrt{x} = \sqrt{y}$.

3. The product of two dissimilar surds is irrational. If possible, let $\sqrt{x} \times \sqrt{y} = ax$, then $xy = a^2x^2$, $y = a^2 x$, $\therefore \sqrt{y} = a\sqrt{x}$; that is, $\sqrt{y}$ and $\sqrt{x}$ may have the same surd factor, which is contrary to the supposition.

4. To extract the square root of a binomial consisting of a rational quantity and a quadratic surd.

Let the given binomial be $a + \sqrt{b}$.

Assume $\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}$; square both sides,

$$a + \sqrt{b} = x + y + 2\sqrt{xy},$$

$$\therefore x + y = a, \quad 2\sqrt{xy} = \sqrt{b}, \quad \text{(Prop. 2.)}$$

$$x^2 + 2xy + y^2 = a^2, \qquad 4xy = b,$$

$$x^2 - 2xy + y^2 = a^2 - b, \qquad \therefore x - y = \sqrt{(a^2 - b)}$$

$$x + y = a$$

$$2x = a + \sqrt{(a^2 - b)}$$

$$2y = a - \sqrt{(a^2 - b)},$$

$$\therefore x = \frac{a + \sqrt{(a^2 - b)}}{2}, \quad y = \frac{a - \sqrt{(a^2 - b)}}{2},$$

$$\therefore \sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y} =$$

$$\pm\sqrt{\frac{a + \sqrt{(a^2 - b)}}{2}} \pm \sqrt{\frac{a - \sqrt{(a^2 - b)}}{2}}$$

Examples.

1 Extract the square root of $7 \pm 4\sqrt{3}$.

Assume $\sqrt{(7 \pm 4\sqrt{3})} = \sqrt{x} \pm \sqrt{y}$,

$$\text{then } 7 \pm 4\sqrt{3} = x + y \pm 2\sqrt{xy},$$

$$\therefore x + y = 7, \quad 2\sqrt{xy} = 4\sqrt{3}.$$

$$x^2 + 2xy + y^2 = 49, \qquad 4xy = 48,$$

$$\therefore x^2 - 2xy + y^2 = 1, \quad \therefore \left.\begin{array}{r} x - y = 1 \\ x + y = 7 \end{array}\right\} \therefore \begin{array}{l} x = 4, \\ y = 3, \end{array}$$

$$\therefore \sqrt{(7 \pm 4\sqrt{3})} = \sqrt{x} \pm \sqrt{y} = 2 \pm \sqrt{3}$$

2. Extract the fourth root of $49 + 20\sqrt{6}$.

Assume $\sqrt{(49 + 20\sqrt{6})} = \sqrt{x} + \sqrt{y}$,

$$\text{then } 49 + 20\sqrt{6} = x + y + 2\sqrt{xy}.$$

$$\therefore x + y = 49, \quad 2\sqrt{xy} = 20\sqrt{6},$$

$$x^2 + 2xy + y^2 = 2401, \qquad 4xy = 2400,$$

$$\therefore x^2 - 2xy + y^2 = 1, \ \therefore \left.\begin{array}{r} x - y = 1 \\ x + y = 49 \end{array}\right\} \therefore \begin{array}{l} x = 25 = 5^2, \\ y = 24 = 4 \times 6, \end{array}$$

$$\therefore \sqrt{(49 + 20\sqrt{6})} = \sqrt{x} + \sqrt{y} = 5 \pm 2\sqrt{6}.$$

Again, assume $\sqrt{(5 \pm 2\sqrt{6})} = \sqrt{x} \pm \sqrt{y}$,

$$\text{then} \quad 5 \pm 2\sqrt{6} = x + y \pm 2\sqrt{xy},$$

$$\therefore x + y = 5, \quad 2\sqrt{xy} = 2\sqrt{6},$$

$$x^2 + 2xy + y^2 = 25, \quad 4xy = 24,$$

$$x^2 - 2xy + y^2 = 1, \quad \left.\begin{array}{r} x - y = 1 \\ x + y = 5 \end{array}\right\} \therefore \begin{array}{l} x = 3, \\ y = 2, \end{array}$$

$$\therefore \sqrt[4]{(49+20\sqrt{6})} = \sqrt{(5 \pm 2\sqrt{6})} = \sqrt{x} \pm \sqrt{y} = \sqrt{3} \pm \sqrt{2}$$

3. Find the eighth root of -1.

The square root of -1 is $\sqrt{-1}$ or $0 + \sqrt{-1}$.

Assume that $\sqrt{x} + \sqrt{y} = \sqrt{(0 + \sqrt{-1})}$.

$$x + y + 2\sqrt{xy} = 0 + \sqrt{-1},$$

$$x + y = 0, \qquad 2\sqrt{xy} = \sqrt{-1},$$

$$\begin{array}{rl} x^2 + 2xy + y^2 &= 0 \\ 4xy &= -1 \\ \hline x^2 - 2xy + y^2 &= 1 \\ x - y &= 1 \\ x + y &= 0 \\ \hline 2x &= 1, \end{array}$$

$$2x = 1, \ \therefore x = \tfrac{1}{2}, \quad \sqrt{x} = \sqrt{\tfrac{1}{2}},$$

$$2y = -1, \therefore y = -\tfrac{1}{2}, \sqrt{y} = \sqrt{-\tfrac{1}{2}},$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{\tfrac{1}{2}} + \sqrt{-\tfrac{1}{2}}.$$

$$\therefore \sqrt{\sqrt{-1}} = \sqrt{\tfrac{1}{2}} + \sqrt{-\tfrac{1}{2}}.$$

Assume $\sqrt{x} + \sqrt{y} = \sqrt{(\sqrt{\tfrac{1}{2}} + \sqrt{-\tfrac{1}{2}})}$,

$$x + y + 2\sqrt{xy} = \sqrt{\tfrac{1}{2}} + \sqrt{-\tfrac{1}{2}},$$

$$x + y = \sqrt{\tfrac{1}{2}}, \qquad 2\sqrt{xy} = \sqrt{-\tfrac{1}{2}},$$

$$x^2 + 2xy + y^2 = \tfrac{1}{2}$$

$$4xy = -\tfrac{1}{2}$$

$$x^2 - 2xy + y^2 = 1$$

$$x - y = 1$$

$$x + y = \sqrt{\tfrac{1}{2}}$$

$$2x = 1 + \sqrt{\tfrac{1}{2}}, \quad \therefore x = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}},$$

$$2y = \sqrt{\tfrac{1}{2}} - 1, \quad \therefore y = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} - \tfrac{1}{2},$$

$$\therefore \sqrt{x} = \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})}, \quad \sqrt{y} = \sqrt{(\tfrac{1}{2}\sqrt{\tfrac{1}{2}} - \tfrac{1}{2})},$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})} + \sqrt{(\tfrac{1}{2}\sqrt{\tfrac{1}{2}} - \tfrac{1}{2})},$$

$$\therefore \sqrt[8]{-1} = \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})} + \sqrt{(\tfrac{1}{2}\sqrt{\tfrac{1}{2}} - \tfrac{1}{2})}$$

$$= \cdot 9239 + \cdot 3827\sqrt{-1} \text{ nearly.}$$

Examples.

1. Extract the square root of $2 + \sqrt{3}$; of $8 + 2\sqrt{7}$; and of $4 - \sqrt{7}$.

Ans. $\sqrt{\dfrac{1}{2}} + \sqrt{\dfrac{3}{2}}$; $1 + \sqrt{7}$; and $\sqrt{\dfrac{7}{2}} - \sqrt{\dfrac{1}{2}}$.

2. Of $8 - 2\sqrt{15}$; $1 - 4\sqrt{-3}$; $1 + \sqrt{(1 - m^2)}$; and $3\sqrt{3} + 2\sqrt{6}$.

Ans. $\sqrt{5} - \sqrt{3}$; $2 - \sqrt{-3}$; $\sqrt{\dfrac{1+m}{2}} + \sqrt{\dfrac{1-m}{2}}$;

and $(1 + \sqrt{2})\sqrt[4]{3}$.

8 Extract the fourth root of $14 + 8\sqrt{3}$; and of $\frac{17}{3} - 4\sqrt{2}$

Ans. $\frac{\sqrt{3}+1}{\sqrt[4]{2}}$; and $(1 - \sqrt{2})\sqrt[4]{\frac{1}{3}}$.

CHAPTER X.

INDETERMINATE COEFFICIENTS.

XXVI. Let $A + Bx + Cx^2 + \ldots = a + bx + cx^2 + \ldots$ be an equation which holds true for any value whatever of x; then the coefficients of like powers of x shall be equal to each other; that is, $A = a$, $B = b$, $C = c$, &c.

For $A - a + Bx - bx + Cx^2 - cx^2 + \&c. \ldots = 0$,

$$\text{or } A \sim a + (B \sim b)x + (C \sim c)x^2 + \ldots = 0.$$

Now, if $A \sim a$ be not equal to 0, let it be equal to some fixed quantity M, then

$$(B \sim b)x + (C \sim c)x^2 + \ldots = M;$$

and $\because$ A and a are invariable quantities, $\therefore A \sim a$ or M is *invariable.* But $\because$ M may have various values dependent upon the variations of x, $\therefore$ M is *variable*; that is, M is both variable and invariable, which is impossible;

$$\therefore A \sim a = 0, \text{ or } A = a.$$

Again, $B \sim b + (C \sim c)x + \ldots = 0$,

$\therefore$ similarly $B = b$, and $C = c$, &c.

If $A + Bx + Cx^2 + \ldots + A'y + B'xy + \ldots + A''y^2 + \ldots =$
$$a + bx + cx^2 + \ldots + a'y + b'xy + \ldots + a''y^2 + \ldots$$
and if some fixed value be given to x while y is variable, it may be shown, as above, that

$$A = a,\ B = b,\ C = c,\ A' = a',\ B' = b',\ C' = c',\ \&c.$$

Examples

1. Resolve $\frac{2x}{(x^2+1)(x^2+3)}$ into its partial fractions

$$\text{Assume } \frac{2x}{(x^2+1)(x^2+3)} = \frac{Ax}{x^2+1} + \frac{Bx}{x^2+3} =$$

$$\frac{Ax^3 + 3Ax + Bx^3 + Bx}{(x^2+1)(x^2+3)},$$

$$\therefore 2x = (A + B)x^3 + (3A + B)x,$$

$$\therefore 2 = 3A + B, \text{ and } (A + B)x^3 = 0, \quad \therefore A = -B,$$

$$\therefore 2 = -3B + B = -2B, \quad \therefore B = -1,$$

$$3A = 2 - B = 2 + 1 = 3, \quad \therefore A = 1$$

$$\text{Hence } \frac{2x}{(x^2+1)(x^2+3)} = \frac{x}{x^2+1} - \frac{x}{x^2+3}.$$

2. Let $y^3 - 3y + x = 0$; find the value of y in a series of ascending powers of x.

$$\left.\begin{array}{rl} \text{Assume } y = & Ax + Bx^3 + Cx^5 + Dx^7 + \&c. \\ \text{then } y^3 = & A^3x^3 + 3A^2Bx^5 + 3A^2Cx^7 + \&c. \\ & \qquad\qquad + 3AB^2x^7 + \&c. \\ -3y = & -3Ax - 3Bx^3 - 3Cx^5 - 3Dx^7 - \&c. \\ +x = & +1x. \end{array}\right\} = 0$$

And, equating coefficients of like powers of x, we have

$$-3A = -1, \text{ or } A = \frac{1}{3}; \; A^3 = 3B, \text{ or } B = \frac{A^3}{3} = \frac{1}{3^4};$$

$$3A^2B = 3C, \text{ or } C = A^2B = \frac{1}{3^6}, \&c.,$$

$$\therefore y = \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \&c$$

If we had assumed $y = Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$, we should have found that the coefficients of the even powers of x would be equal to 0.

Examples.

1. Find the fractions whose sum is $\dfrac{x^2}{(x^2-1)(x-2)}$.

Ans. $\dfrac{4}{3(x-2)} - \dfrac{1}{2(x-1)} + \dfrac{1}{6(x+1)}$.

2. Expand $\dfrac{1+2x}{1-x-x^2}$ in a series of ascending powers of x

Ans. $1 + 3x + 4x + 7x^3 + 11x^4 +$ &c.

3. Prove that $\surd(1+x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 -$ &c

4. Prove that $1^2 + 2^2 + 3^2 + 4^2 +$ &c. $+ n^2 =$

$$\frac{n(n+1)(2n+1)}{6}.$$

CHAPTER XI.

LOGARITHMS.

XXVII. If $a^x = n$, x is called the *logarithm* of the number n to the base a. In the common system of logarithms, $a = 10$, and 0, 1, 2, 3, 4 ... x, are the logarithms of the numbers 1, 10; 100, 1000 ... 10^x; that is, the numbers being in *geometric* progression, their logarithms are in *arithmetic* progression.

Logarithms are calculated by finding x from the equation $10^x = n$, in series involving the powers of x, and are published in books of Logarithmic Tables.

Propositions.

1. The sum of the logarithms of two or more numbers is the logarithm of their product.

Let $x = \log n$, and $y = \log n'$; then, a being the base,

$$a^x = n, \ a^y = n'; \ \therefore a^x a^y = nn', \text{ or } a^{x+y} = nn'.$$

And similarly for more numbers.

2. The difference of the logarithms of two numbers is the logarithm of their quotient.

$$\text{For } a^x = n, \ a^y = n'; \ \therefore \frac{a^x}{a^y} = \frac{n}{n'}, \text{ or } a^{x-y} = \frac{n}{n'}.$$

3. The logarithm of $n^m = m \log n$.

$$\text{For} \because n = a^x, \quad \therefore n^m = a^{mx};$$

$$\therefore \log n^m = mx = m \log n.$$

4. The logarithm of $\sqrt[m]{n} = \dfrac{\log n}{m}$.

$$\text{For} \because n = a^x, \quad \therefore n^{\frac{1}{m}} = a^{\frac{x}{m}};$$

$$\therefore \log n^{\frac{1}{m}} = \frac{x}{m} = \frac{\log n}{m}.$$

Hence the *multiplication* and *division* of numbers may be performed by the *addition* and *subtraction* of logarithms, and the *involution* or *evolution* of numbers, by *multiplying* or *dividing* their logarithms by the indices of the powers or roots required.

Examples.

1. If $x = ab$, then $\log x = \log a + \log b$.
2. If $y = 6ac$, then $\log y = \log 6 + \log a + \log c$.
3. If $x = \dfrac{m}{r}$, then $\log x = \log m - \log r$
4. If $y = a^3$, then $\log y = 3 \log a$.
5. If $x = \sqrt[4]{c}$, then $\log x = \dfrac{1}{4} \log c$.

6. If $S = a \cdot \dfrac{r^n - 1}{r - 1}$, find a logarithmic expression for the value of n.

$$S(r-1) = ar^n - a, \quad ar^n = S(r-1) + a,$$

$$r^n = \frac{S(r-1)+a}{a}, \ \therefore \ n \log r = \log\{S(r-1)+a\} - \log a,$$

$$\therefore n = \frac{\log\{S(r-1)+a\} - \log a}{\log r}$$

7. If $5^x = 400$, find x.

$$x \log 5 = \log 400,$$

$$\therefore x = \frac{\log 400}{\log 5} = \frac{2{\cdot}602060}{{\cdot}698970} = 3{\cdot}72 \text{ nearly.}$$

The logarithms of 400 and 5 are taken from the tables of Brigg's Logarithms.

The number standing before the decimal part of a logarithm is called the *characteristic*, and it is always less by unity than the number of digits in the integral part of the number: thus the number 400 consisting of three digits, the characteristic of its logarithm is 2. Hence, in tables of logarithms, it is usual to omit the characteristic.

The logarithm of 1752 is 3·243534.
" of 175·2 is 2·243534.
" of 17·52 is 1·243534.
" of 1·752 is 0·243534.
" of ·1752 is − 1 + ·243534.
" of ·01752 is − 2 + ·243534.
&c. &c.

When the characteristic is negative, it is usually written with the minus sign *above* it, and the + connecting it with the decimal omitted; thus

$$\log {\cdot}1752 = \bar{1}{\cdot}243534, \ \log {\cdot}01752 = \bar{2}{\cdot}243534, \ \&c.$$

CHAPTER XII.

BINOMIAL THEOREM.

XXVIII. The Binomial Theorem enables us to raise a binomial to any power without the process of repeated multiplication.

Prop. Let it be required to find the nth power of the binomial $a + x$.

$$a + x = a\left(1 + \frac{x}{a}\right), \quad \therefore (a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n.$$

Assume $\frac{x}{a} = y$, and that

$$(1 + y)^n = A + By + Cy^2 + Dy^3 + \ldots Py^p + \&c. \ldots (1),$$

the coefficients A, B, C, D, &c., being independent of y.

Squaring both sides of this equation, we have

$$\left.\begin{aligned}(1+y)^{2n} = A^2 + 2ABy + 2ACy^2 + 2ADy^3 + 2AEy^4 + \&c.\\ + B^2y^2 + 2BCy^3 + 2BDy^4 + \&c.\\ + C^2y^4 + \&c.\end{aligned}\right\}\ldots (2)$$

Now $\because (1 + y)^{2n} = (1 + 2y + y^2)^n = \{1 + (2y + y^2)\}^n$,

$\therefore$ by assumption (1),

$$\begin{aligned}(1 + y)^{2n} &= A + B(2y + y^2) + C(2y + y^2)^2 + D(2y + y^2)^3 + \&c.\\ &= \left.\begin{aligned}A + 2By + By^2 + 4Cy^2 + Cy^4 + \&c.\\ + 4Cy^3 + 8Dy^3 + 16Ey^4 + \&c.\end{aligned}\right\}\ldots\ldots (3).\end{aligned}$$

And $\because$ the series (2) and (3) are each $= (1 + y)^{2n}$,

$\therefore A^2 + 2ABy + (2AC + B^2)y^2 + (2AD + 2BC)y^3 + \ldots \&c. =$
$A + 2By + (B + 4C)y^2 + (4C + 8D)y^3 + \ldots \&c.$

Hence, by Art. XXVI., $A^2 = A$, $\quad \therefore A = 1$;

$$2AB = 2B, \quad \therefore B = B;$$

$$2AC + B^2 = B + 4C, \quad \therefore C = \frac{B^2 - B}{2} = \frac{B(B-1)}{1.2};$$

$$2AD + 2BC = 4C + 8D,$$

$$\therefore D = \frac{2C(B-2)}{6} = \frac{B(B-1)(B-2)}{1.2.3}; \ \&c.$$

Hence, $(1+y)^n = 1 + By + \frac{B(B-1)}{1.2}y^2 + \frac{B(B-1)(B-2)}{1.2.3}y^3 + \&c. \ldots (4).$

Putting My^2 for all the terms after By, we have

$$(1+y)^n = 1 + By + My^2, \quad n\log(1+y) = \log(1 + By + My^2), \text{ Art. XXVII.},$$

and, if we assume $\log(1+y) = a + by + cy^2 + \&c.$, a, b, c, &c., being independent of y; then

$$\log\{1 + (By + My^2)\} = a + b(By + My^2) + c(By + My^2)^2 + \&c.$$

$$\therefore na + nby + ncy^2 + \&c. = a + bBy + bMy^2 + \&c.,$$

$$\therefore \text{by Art. XXVI.}, \ na = a, \quad \therefore a = 0, \quad nb = bB, \quad \therefore B = n.$$

Hence, substituting n for B in (4), we have

$$(1+y)^n = 1 + ny + \frac{n(n-1)}{1.2}y^2 + \frac{n(n-1)(n-2)}{1.2.3}y^3 + \&c.,$$

$$\text{or, } \left(1 + \frac{x}{a}\right)^n = 1 + n\frac{x}{a} + \frac{n(n-1)}{1.2}\frac{x^2}{a^2} + \frac{n(n-1)(n-2)}{1.2.3}\frac{x^3}{a^3} + \&c.,$$

$$\therefore (a+x)^n = a^n\left(1 + \frac{x}{a}\right)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \&c.,$$

which is the Binomial Theorem of Sir Isaac Newton.

Cor. 1. If x be negative, its even powers will be positive, and its odd powers negative; hence

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \&c.$$

Cor. 2. If $a = 1$, $(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c$

$$\because (a + b + c)^n = \{(a + b) + c\}^n,$$
$$\text{and } (a + b + c + d)^n = \{(a + b) + (c + d)\}^n,$$

the Binomial Theorem may be applied to the expansion of powers of a polynomial.

The above proof is general, and consequently applies to the cases of n being an integer, or a fraction, positive or negative.

On inspecting this theorem, it appears that

The powers of a decrease and those of x increase by unity in each succeeding term.

The *first* term is the nth power of the first term of the given binomial.

The *second* term is found by multiplying the index of the first by the first with its index diminished by unity, and also by the second term of the binomial.

The coefficient of any succeeding term is found by multiplying the coefficient and index of a in the preceding term together, and dividing by the number of terms already set down.

In applying this theorem, it will appear that

When n is an integer, the number of terms in the expansion is $n + 1$; and the coefficients of any two terms equidistant from the extremes are equal to each other.

When n is a fraction the series does not terminate.

Examples.

1. Expand $(a + x)^6$.

$$(a + x)^6 = a^6 + 6a^5x + \frac{6 . 5}{2} a^4x^2 + \frac{6 . 5 . 4}{2 . 3} a^3x^3$$
$$+ \frac{6 . 5 . 4 . 3}{2 . 3 . 4} a^2x^4 + \frac{6 . 5 . 4 . 3 . 2}{2 . 3 . 4 . 5} a x^5 + \frac{6 . 5 . 4 . 3 . 2 . 1}{2 . 3 . 4 . 5 . 6} x^6$$
$$= a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6.$$

Practically, the coefficients after the second term are found thus,

$$\frac{6 . 5}{2} = 15, \quad \frac{15 . 4}{3} = 20$$

and after the middle term they may be written in an inverse order.

2. $(a-x)^5 = a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$
3. $(1-x)^7 = 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7.$
4. $(2x+1)^5 = 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1.$
5. $\frac{1}{(c+x)^2} = \frac{1}{c^2}\left(1 - \frac{2x}{c} + \frac{3x^2}{c^2} + \frac{4x^3}{c^3} + \text{\&c.}\right).$
6. $\sqrt[3]{(1-x^3)} = 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} - \text{\&c.}$

EXPONENTIAL THEOREM.

XXIX. $\because a = 1 + \overline{a-1}, \quad \therefore a^x = (1 + \overline{a-1})^x =$

$$\{(1 + \overline{a-1})^n\}^{\frac{x}{n}}.$$

Hence, by the Binomial Theorem (Cor. 2),

$$a^x = \left\{1 + n(a-1) + \frac{n(n-1)}{2}(a-1)^2 + \right.$$

$$\left. \frac{n(n-1)(n-2)}{2 \cdot 3}(a-1)^3 + \text{\&c.}\right\}^{\frac{x}{n}} =$$

$$\left\{1 + \left[(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.}\right] n + \right.$$

$$\left. Bn^2 + Cn^3 + \text{\&c.}\right\}^{\frac{x}{n}},$$

where B, C, &c. contain powers of $(a-1)$ only.

Let now $(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.} = A$, then

$$a^x = \{1 + An + Bn^2 + Cn^3 + \text{\&c.}\}^{\frac{x}{n}} =$$

$$1 + \frac{x}{n}(An + Bn^2 + \text{\&c.}) +$$

$$\frac{\frac{x}{n}\left(\frac{x}{n} - 1\right)}{2}(An + Bn^2 + \text{\&c.})^2 + \text{\&c} =$$

$$1 + x(A + Bn + \&c.) + \frac{x(x-n)}{2}(A + Bn + \&c.)^2 + \&c.$$

Now since n may have any value whatever, let $n = 0$, then

$$a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c$$

If e be that value of a which makes $A = 1$,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c., \text{ the Exponential Theorem.}$$

If $x = 1$,

$$e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. = 2{\cdot}71828, \&c., \text{ which is}$$

the base of the system of logarithms used by Napier, the inventor of logarithms.

XXX. $\because a + b = \dfrac{a}{1 - \dfrac{b}{a+b}}$,

$$\therefore (a+b)^n = \frac{a^n}{\left(1 - \dfrac{b}{a+b}\right)^n} = a^n\left(1 - \frac{b}{a+b}\right)^{-n}.$$

$\therefore (a+b)^n =$

$$a^n\left\{1 + n\cdot\frac{b}{a+b} + \frac{n(n+1)}{2}\cdot\left(\frac{b}{a+b}\right)^2 + \&c.\right\} \text{(Art. XXVIII.)}$$

This is Colson's Theorem.

CHAPTER XIII.

VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

XXXI. The different arrangements which can be made of any number of quantities, taking a certain number at a time, are called their *Variations*.

Thus, if a, b, c be taken two together, their variations will be ab, ba, ac, ca, bc, cb.

If all the quantities are taken together, their variations are called *Permutations*.

Thus the permutations of a, b, c are abc, acb, bac, bca, cab, cba.

XXXII. The number of variations of n different things, taken r together, is $n(n-1)(n-2)\ldots\{n-(r-1)\}$.

Let a, b, c, d, &c., be the n things; then the number of variations which can be made, taking them singly, is n.

Let $n-1$ of these things, namely b, c, d, &c., be taken singly, then the number of their variations is $n-1$: and if a be placed before each, we shall have $n-1$ variations of n things, taken two together, in which a stands first; similarly we shall have $n-1$ such variations, in which b stands first; and similarly for all the n things, hence there will be on the whole $n(n-1)$ variations of n things taken *two* together.

Again, taking $n-1$ of these things, namely b, c, d, &c., their variations, taken *two* together, will be $n(n-1)(n-2)$; and proceeding as before, there will be on the whole $(n-1)(n-2)$ variations of n things taken *three* together.

Similarly their variations, taken *four* together, will be $n(n-1)(n-2)(n-3)$. Hence, if V_1, V_2, V_3, &c., V_r denote the variations of n things, taken 1, 2, 3, &c., r together, we have

$$V_1 = n, \quad V_2 = n(n-1), \quad V_3 = n(n-1)(n-2), \text{ \&c.}$$
$$V_r = n(n-1)(n-2)(n-3)\ldots\{n-(r-1)\}.$$

Cor. $\because$ the permutations (p) of n things are their variations, taken *all together*; by writing n for r we shall have

$$p = n(n-1)(n-2)\ldots\{n-(n-2)\}\{n-(n-1)\}$$
$$= n(n-1)(n-2)\ldots 2 \cdot 1 = 1 \cdot 2 \cdot 3 \ldots n.$$

XXXIII. When a and b are different, their permutations are ab, ba, but when $a = b$, they become aa.

Let a recur p times, b, q times, c, r times, &c., and P be the number of permutations required. Then if all the a's be changed into different letters, they will form $1 \cdot 2 \cdot 3 \ldots p$

permutations, and out of *each* of the P permutations we should form $1 \,.\, 2 \,.\, 3 \ldots\ldots p$ permutations, so that there would be P times $1 \,.\, 2 \,.\, 3 \ldots p$ permutations. Similarly if all the *b*'s be changed to different letters, they would form $1 \,.\, 2 \,.\, 3 \ldots q$ permutations, and therefore there would be $P \,.\, (1 \,.\, 2 \,.\, 3 \ldots\ldots p \,.\, 1 \,.\, 2 \,.\, 3 \ldots\ldots q)$ permutations. Now, when all the quantities have become different, the number of permutations is $1 \,.\, 2 \,.\, 3 \ldots n$. (Art. XXXII., *Cor.*)

$$\therefore P \,.\, (1 \,.\, 2 \,.\, 3 \ldots p \,.\, 1 \,.\, 2 \,.\, 3 \ldots q \,.\, 1 \,.\, 2 \,.\, 3 \ldots r \,.\, \&c.) = 1 \,.\, 2 \,.\, 3 \ldots n,$$

$$\therefore P = \frac{1 \,.\, 2 \,.\, 3 \ldots\ldots n}{1 \,.\, 2 \,.\, 3 \ldots p \,.\, 1 \;\, 2 \,.\, 3 \ldots\ldots q \,.\, 1 \,.\, 2 \,.\, 3 \ldots r \,.\, \&c.}.$$

XXXIV. Suppose there are n things, a, b, c, d, &c., and that they may be repeated; then, if they be taken two together, we shall have n variations where a stands first, n where b stands first, n where c stands first, &c. And since there are n things, the entire numbers of variations will be $n \times n$, or n^2.

If they be taken three together, since each of them may stand before each variation, the entire number of variations will be $n \times n^2$, or n^3.

In the same manner, if they be taken n together, the entire number of variations will be n^n.

And if it be required to determine the total number of variations with repetitions of n things taken one at a time, two at a time, &c., and all together, we must take the sum of the geometric series $n + n^2 + n^3 + \ldots n^n$

XXXV. The different collections that can be made of a number of things, taking a certain number together without regarding their order, are called their *Combinations*.

Thus the combinations of a, b, c, taken two together, are ab, ac, bc.

Each combination will supply as many corresponding variations as the number of things it contains admits of permutations.

Each combination of r things supplies $1 \,.\, 2 \,.\, 3 \ldots r$ variations of r things; hence, if C_r be the number of combinations of n things, taken r together,

$$C_r . (1 . 2 . 3 \ldots r) = V_r = n(n-1)(n-2) \ldots \{n-(r-1)\},$$

$$\therefore C_r = \frac{n(n-1)(n-2) \ldots \{n-(r-1)\}}{1 . 2 . 3 \ldots r}.$$

Cor. If 1, 2, 3, 4, &c., be successively put for r, we shall have

$$C_1 = \frac{n}{1},\ C_2 = \frac{n(n-1)}{1 . 2},\ C_3 = \frac{n(n-1)(n-2)}{1 . 2 . 3};\ \&c.$$

$$\therefore C_1 + C_2 + C_3 + \&c. = n + \frac{n(n-1)}{1 . 2} + \frac{n(n-1)(n-2)}{1 . 2 . 3} + \&c.$$

But $(1+x)^n = 1 + nx + \frac{n(n-1)}{1 . 2}x^2 + \frac{n(n-1)(n-2)}{1 . 2 . 3}x^3 + \&c.$ (Art. XXVIII.)

$$\therefore (1+1)^n = 2^n = 1 + n + \frac{n(n-1)}{1 . 2} + \frac{n(n-1)(n-2)}{1 . 2 . 3} + \&c.$$

$$\therefore 2^n - 1 = n + \frac{n(n-1)}{1 . 2} + \frac{n(n-1)(n-2)}{1 . 2 . 3} + \&c.$$

$\therefore C_1 + C_2 + C_3 + \&c. = 2^n - 1 =$ the sum of all the combinations that can be made of n things taken 1, 2, 3, &c., n together.

XXXVI. The number of combinations of n sets of things, containing respectively p, q, r, &c., things, one being taken out of each set for each combination, is equal to the continued product of p, q, r, &c.

First, if there be two sets, containing p and q things respectively, the number of combinations will be pq.

Then, if there be another set, containing r things, each of them being combined with pq combinations, the number of combinations will be pqr, &c.

Cor. If $p = q = r =$ &c., the number of combinations $= p^n$.

XXXVII. If of n things r be taken, $n - r$ things will be left, and for every different set of r things taken, there will be a different set containing $n - r$ remaining; therefore, the *number* of the former sets will constantly be the same as that of the latter.

Hence, the number of combinations of n things taken r together is the same as the number of combinations of n

things taken $n - r$ together. This may be expressed thus, ${}^nC_r = {}^nC_{n-r}$.

XXXVIII. To find the number of combinations of two sets of things containing respectively m and n things, by taking r out of one set and s out of the other for each combination.

$$ {}^mC_r \times {}^nC_s = \frac{m(m-1)\ldots(m-r+1)}{1.2\ldots r} \times \frac{n(n-1)(n-s+1)}{1.2\ldots s}. $$

(Art. XXXV.)

XXXIX. To find what number r out of n things must be taken together, so that the number of combinations formed may be a maximum.

Since C_r is obtained by multiplying C_{r-1}, by $\dfrac{n-r+1}{r}$, or $\dfrac{n+1}{r} - 1$, the quantities C_1, C_2, &c., will increase continually, each term upon the preceding, so long as their factor > 1; hence C_r will be the greatest for the greatest value of r which allows of this, or if $n + 1 > 2r$, that is, when r is the integer next $< \frac{1}{2}(n + 1)$.

If n be even, $r = \frac{1}{2}n$, if n be odd, and $\therefore \frac{1}{2}(n + 1)$ an integer, $r = \frac{1}{2}(n + 1) - 1 = \frac{1}{2}(n - 1)$; but in this case, since $C_r = C_{n-r}$ (Art. XXXVII.), the number taken $\frac{1}{2}(n - 1)$ together = number taken $n - \frac{1}{2}(n - 1)$ or $\frac{1}{2}(n + 1)$ together, and $\therefore r$ may be either $\frac{1}{2}(n - 1)$ or $\frac{1}{2}(n + 1)$.

Examples.

1. How many changes can be rung with 7 bells out of 10?

By Variations, Art. XXXII.,

$$ V_r = n(n-1)(n-2)\ldots\{n-(r-1)\}; $$

and $\because$ there are 10 bells, $n = 10$,

and $\because$ they are taken 7 at a time, $r = 7$, and $r - 1 = 6$,

$$ \therefore n - (r - 1) = 10 - 6 = 4. $$

Hence $V_7 = 10 . 9 . 8 . 7 . 6 . 5 . 4 = 604800$ changes.

2. How often can 6 boys change their places in a class, so as not to preserve the same order?

By Permutation, Art. XXXII., *Cor.*, $p = 1 . 2 . 3 \ldots n$,

$$ \because p = 1 . 2 . 3 . 4 \;\; 5 \;\; 6 = 720 \text{ times}. $$

3. In how many ways may the word enunciation be written?

By Permutation, Art. XXXIII., $P = \frac{1.2.3\ldots n}{1.2.p.1.2..q.1.2..r}$;

and $\because$ there are 11 letters, of which 3 are n's and 2 are i's, $\therefore n = 11$, $p = 3$, $q = 2$.

$$\text{Hence, } P = \frac{1.2.3.4.5.6.7.8.9.10.11}{1.2.3.1.2} = 3326400 \text{ ways.}$$

4. Into how many different triangles may a decagon be divided by drawing lines from the angular points?

The number of triangles will be equal to the number of lines that can be drawn by connecting 7 at a time of the 10 angles with each angle; that is, it will be equal to the number of combinations of 10 angles taken 7 together. Combination (Art. XXXV.),

$$C = \frac{n(n-1)(n-2)\ldots\{n-(r-1)\}}{1.2.3\ldots r},$$

$$= \frac{10.9.8.7.6.5.4}{1.2.3.4.5.6.7} = 120.$$

5. The total number of combinations of $2n$ things : total number of n things :: 129 : 1; find n.

Combinations (Art. XXXV., *Cor.*), $2^n - 1$; write $2n$ for n, then

$2^{2n} - 1$ = total number of combinations of $2n$ things,

$2^n - 1 =$ „ „ n „

$$\frac{2^{2n}-1}{2^n-1} = \frac{129}{1}, \quad \text{or } \frac{(2^n+1)(2^n-1)}{2^n-1} = \frac{129}{1},$$

$2^n + 1 = 129$, $2^n = 128$, $2^n = 2^7$, $\therefore n = 7$. Ans.

6. How many different collections and words may be formed of 5 letters of the alphabet, 2 vowels and 3 consonants being taken together?

Combinations (Arts. XXXVIII., XXXV.),

$${}^mC_r \times {}^nC_s = \frac{m(m-1)\ldots(m-r+1)}{1.2\ldots r} \times \frac{n(n-1)\ldots(n-s+1)}{1.2\ldots s}.$$

$$\left.\begin{array}{ll}\text{Here } m = 19 \text{ consonants}, & n = 5 \text{ vowels} \\ \phantom{\text{Here }} r = \ 3 \quad ,, & s = 2 \quad ,, \end{array}\right\} \begin{array}{l} m - r + 1 = 17, \\ n - s + 1 = \ 4, \end{array}$$

$$\therefore {}^mC_r \times {}^nC_s = \frac{19.18.17}{1.2.3} \times \frac{5.4}{1.2} = 969 \times 10 = 9690 \text{ collections.}$$

But since each collection consists of 5 letters, each may be permuted $5.4.3.2.1$ different ways; for (Permutation, Art. XXXII., *Cor.*)

$$p = n(n-1).(n-2) \ldots 3.2.1.$$

Here $n = 5$,

$\therefore p = 5.4.3.2.1 = 120$ permutations of the 9690 collections,

$$\therefore 9690 \times 120 = 1162800 \text{ words. Ans.}$$

7. A telegraph has m arms, and each arm is capable of n distinct positions; find the total number of signals that can be made with the telegraph.

Combination (Art. XXXV., *Cor.*). First find the number of combinations of m arms taken 1 at a time, 2 at a time, &c.,

$${}^mC_1 = m, \ {}^mC_2 = \frac{m(m-1)}{1.2}, \ {}^mC_3 = \frac{m(m-1)(m-2)}{1.2.3}, \text{ \&c.};$$

$$\therefore {}^mC_1 + {}^mC_2 + {}^mC_3 + \text{\&c.} = m + \frac{m(m-1)}{1.2} + \frac{m(m-1)(m-2)}{1.2.3}$$

$+$ &c. $=$ number of combinations of the m arms, taken at first singly, then 2 at a time, then 3 at a time, &c. But as each arm is capable of n distinct positions, the entire number of combinations with changes of position $=$

$$mn + \frac{m(m-1)}{1.2}n^2 + \frac{m(m-1)(m-2)}{1.2.3}n^3 + \text{\&c.} =$$

$(1+n)^m - 1$. Art. XXVIII., *Cor.* 2.

7. How many changes may be rung on 5 bells? Ans. 120.

8. How many different numbers can be made out of one unit, two 2's, three 3's, and four 4's, supposing all the figures to be in every number? Ans. 12600.

9 How many changes can be rung with 4 bells out of 8? Ans. 1680.

10. Given the number of variations of $2n + 1$ things, $n - 1$ together : number of variations of $2n - 1$, n together, $:: 3 : 5$; determine n. Ans. 4.

11. How many different hands may a person hold at the game of whist; that is, how many changes can be made of 13 cards out of 52? Ans. 635013559600.

12. A company of soldiers consists of 50 men, and 4 of them are selected every night to mount guard; on how many nights can a different selection of the 4 sentinels be made? Ans. 230300.

13. What is the total number of combinations of 16 things, taken 1, 2, 3, &c., at a time? Ans. 65535.

14. The number of variations of $m + n$ things, 2 together, is 56, and of $m - n$ things is 12; find the number of combinations of m things, n together. Ans. 15.

15. Find the number of permutations with repetitions (that is, allowing things which recur to be combined as if they were different) of 12 things, taken 5 together. Ans. 248832.

16. Show that the number of different combinations of n things taken 1, 2, 3, &c., n together, of which p are of one sort, q of another, r of another, &c., is $(p + 1)\ (q + 1)\ (r + 1) \ldots 1$.

17. There are 4 companies of soldiers, consisting of 40, 42, 45 and 50 men respectively; how many selections of 4 men can be made by taking one out of each company? Ans. 3780000.

CHAPTER XIV.

COMPOUND INTEREST AND ANNUITIES.

XL. Let P be the *principal*, or sum lent,

M the *amount*, or sum of principal and interest,

r the interest of £1 for the first period of time,

$\mathrm{R} = 1 + r$, the amount of £1 for that period,

n the number of periods.

Then, at the end of the first period R will be the principal, and $\because$ £1 invested amounts to R pounds,

$$\therefore 1 : R :: R : R^2 = \text{amount of £1 in 2 periods,}$$

similarly $1 : R^2 :: R : R^3 =$ „ 3 „

and so on. $\therefore R^n =$ amount of £1 in n periods of time of equal length; and when £P is the given principal, the amount will be P times as great as when £1 is the principal.

$$\therefore M = PR^n, \log M = \log P + n \log R \text{ (Art. XXVII.).}$$

By this last equation, any three of the quantities M, P, n, R being given, the fourth may be found.

XLI. Let A be an annuity, or sum of money annually payable, R the amount of £1 in one year; then, A is the amount of the annuity at the end of the first year, $1 : A :: R : RA =$ its amount at the end of the second year, $\because A + RA = A(1 + R)$ is the sum due at the end of the second year; similarly $1 : A(1 + R) :: R : A(R + R^2)$ is the amount of the two payments at the end of the third year, $\therefore A + A(R + R) = A(1 + R + R^2)$ is the sum due at the end of the third year; similarly $A(1 + R + R^2 + R^3)$ is the sum due at the end of the fourth year; and generally

$$A(1 + R + R^2 + \ldots R^{n-1}) = A \cdot \frac{R^n - 1}{R - 1} = M, \text{ the } \textit{amount}$$

of the annuity, at compound interest, in n years.

$$\log M = \log A + \log(R^n - 1) - \log(R - 1).$$

XLII. Let V be the *present value* of the annuity A, then

$$VR^n = A \cdot \frac{R^n - 1}{R - 1}, \quad \therefore V = A \cdot \frac{1 - R^{-n}}{R - 1},$$

$$\log V = \log A + \log(1 - R^{-n}) - \log(R - 1).$$

Cor. If the annuity be *perpetual*, n is infinite,

$$\therefore R^{-n} = \frac{1}{R^n} = 0,$$

and $V = \dfrac{A}{R - 1}$ = the present value of a *perpetual annuity* of £A per annum.

XLIII. Let the annuity A commence after m years, and continue for n years, then its present value will be the difference between its present value for $m + n$ years, and its present value for m years, or

$$V = A\,\frac{1 - R^{-(m+n)}}{R - 1} - A\,\frac{1 - R^{-m}}{R - 1} = \frac{AR^{-m} - AR^{-m-n}}{R - 1}$$

$$= \frac{AR^{-m}(1 - R^{-n})}{R - 1}$$

$$\log V = \log A - m \log R + \log(1 - R^{-n}) - \log(R - 1).$$

Cor. If the annuity be perpetual, n is infinite, $\therefore R^{-n} = 0$, and $V = \dfrac{AR^{-m}}{R - 1}$ = the present value of a *deferred annuity*, to continue for ever.

XLIV. Let N denote any number, whole, fractional, or mixed; let λN denote its logarithm, and κN the complement of that logarithm, so that $\lambda \dfrac{1}{N} = -\lambda N = \kappa N$; then the formulæ adapted to logarithmic computation in the last four articles may be put into a more convenient shape by writing λ for log, and κ for $-$ log.

Examples.

1. What sum laid up now and improved at compound interest during 12 years, will amount to 5000*l.*, the rate of interest being $4\frac{1}{2}$ per cent., the interest being payable yearly?

By Art. XL. $\quad \log P = -n \log R + \log M$, or

$$\lambda P = n\kappa R + \lambda M$$
$$= 12\kappa(1{\cdot}045) + \lambda 5000$$

Now $\kappa\, 1{\cdot}045$	$= \bar{1}{\cdot}9808837$
	$\times\, n = 12$
	$\bar{1}{\cdot}7706044$
$\lambda\, 5000$	$= 3{\cdot}6989700$
$\lambda\, P$	$= 3{\cdot}4695744$

$\therefore P = 2948{\cdot}31836 = £2948\ 6s.\ 4\frac{1}{2}d.$ Ans.

2. To what sum will an annuity of £100 amount in 20 years, at compound interest, at 4½ per cent., interest payable yearly?

By Art. XLI.

$$\begin{aligned} \log M &= \log(R^n - 1) + \log A - \log(R - 1), \text{ or} \\ \lambda M &= \lambda(R^n - 1) + \lambda A + \varkappa(R - 1), \\ &= \lambda(1 \cdot 045^{20} - 1) + \lambda 100 + \varkappa \cdot 045 \end{aligned}$$

$$\begin{array}{lr} \text{Now } \lambda\, 1 \cdot 045 = & \cdot 0191163 \\ & 20 \\ \hline & \cdot 3823260 \\ \text{the number to which is} & 2 \cdot 4117150 \\ & -1 \\ \hline & 1 \cdot 4117150 \\ \text{the log of which is} & \cdot 1497470 \\ \lambda\, 100 = & 2 \\ \hline & 2 \cdot 1497470 \\ \varkappa \cdot 045 & 1 \cdot 3467875 \\ \hline \lambda M = & 3 \cdot 4965345, \end{array}$$

$$\therefore M = 3137 \cdot 144 = £3137\ 2s.\ 10\tfrac{1}{2}d. \quad \text{Ans.}$$

If the interest were payable half-yearly, n would be 40, r would be $\frac{\cdot 045}{2} = \cdot 0225$, $\therefore$ R would be $1 \cdot 0225$.

If the interest were payable quarterly, n would be 80, and R $1 \cdot 01125$.

3. What is the value of the lease of an estate for 21 years, the clear annual rental being £135, allowing 5 per cent. compound interest?

By Art. XLII.

$$\begin{aligned} \log V &= \log(1 - R^{-n}) + \log A - \log(R - 1), \text{ or} \\ \lambda V &= \lambda(1 - R^{-n}) + \lambda A + \varkappa(R - 1) \\ &= \lambda(1 - 1 \cdot 05^{-21}) + \lambda 135 + \varkappa \cdot 05 \end{aligned}$$

$$\text{Now } \lambda R^{-n} = \lambda \frac{1}{R^n} = \kappa R^n = \kappa\, 1{\cdot}05^{21} = \bar{1}{\cdot}9788107$$

$$\begin{array}{r} 21 \\ \hline 9788107 \\ 19576214 \\ \hline 21 \\ \hline \bar{1}{\cdot}5550247, \end{array}$$

$$\therefore R^{-n} = {\cdot}3589422, \qquad 1 - R^{-n} = {\cdot}6410578$$

$$\begin{array}{ll} \lambda\, {\cdot}6410578 & = \bar{1}{\cdot}8068973 \\ \lambda\, {\cdot}135 & = 2{\cdot}1303338 \\ \kappa\, {\cdot}05 & = 1{\cdot}3010300 \\ \hline \lambda\, V & = 3{\cdot}2382611 \end{array}$$

$\therefore V = 1730{\cdot}8565 =$ £1730 17*s.* $1\frac{1}{2}$*d.* Ans.

4. Find the compound interest of £800 for 9 years at 5 per cent. per annum, the interest being payable yearly.
Ans. £441 1*s.* 3*d.*

5. Find the amount of an annuity of £356 per annum, payable half-yearly for 9 years, allowing compound interest at 6 per cent. per annum. Ans. £4167 15*s.* $4\frac{1}{4}$*d.*

6. What is the present value of an annuity of £70 per annum, payable quarterly for five years, allowing compound interest at 5 per cent. per annum? Ans. £307 19*s.* $8\frac{1}{2}$*d.*

7. Find the present value of a deferred annuity of £1000, to commence after the expiration of 5 years, and then to continue for 20 years, allowing compound interest at 5 per cent Ans. £9764 9*s.* $4\frac{1}{4}$*d.*

8. If a lease for $55\frac{1}{4}$ years cost £100, what annual rent ought the purchaser to receive, that he may get $5\frac{1}{2}$ per cent for his money? Ans. £5 16*s.*

9. What is the difference between the value of a freehold estate, or perpetual annuity of £100 per annum, and that of a leasehold estate of £100 per annum, to continue 60 years?

Ans. The freehold is worth £107 1*s*. $4\frac{3}{4}$*d*. more than the leasehold.

10. What sum ought to be paid for the reversion of an annuity of £50 for 14 years after the next 7, that the purchaser may make 5 per cent. of his money? Ans. £351 10*s*. nearly.

11. The sum of £518 6*s*. being placed out at compound interest for 3 years, amounts to £600; find the rate of interest. Ans. 5 per cent.

12. In what time will a sum of money *double* itself, at 4 per cent. compound interest? Ans. In 17·6 years.

13. Suppose a person to place out annually the sum of £20 for 40 successive years, what would the whole amount to at the end of that time, at 5 per cent. compound interest?

Ans. £2416.

14. Find the present value of an annuity of £40, to continue 5 years, allowing compound interest at 5 per cent.

Ans. £173 3*s*. 7*d*.

15. What must be paid for an estate whose annual rental is £79 4*s*. that the purchaser may make $4\frac{1}{2}$ per cent. of his money? Ans. £1760.

16. A person places P pounds at interest, and adds to his capital at the end of every year $\frac{1}{m}$th part of the interest for that year; what is the amount at the expiration of n years.

Ans. $P\left(\frac{r}{m}+r+1\right)^n$.

17. Find the present value of an annuity of £20, to commence 10 years hence, and then to continue 11 years, allowing $3\frac{1}{2}$ per cent. compound interest. Ans. £118 7*s*. $3\frac{1}{2}$*d*.

18. A person leaves to his two sons, A and B, equal shares of an estate producing £1000 per annum, but A proposes that B shall take both shares, and allow him an equivalent annuity for 20 years; what annuity ought B to allow A, reckoning interest at $4\frac{1}{2}$ per cent.? Ans. £854 2*s*. 4*d*

APPENDIX,

CONTAINING MISCELLANEOUS INVESTIGATIONS AND EXAMPLES.

I. In the equation

$$ar^n + br^{n-1} + cr^{n-2} + \ldots\ldots pr^2 + qr + s = N,$$

r and N are given numbers: it is required to find the numerical values of the coefficients a, b, c, &c., and also the value of the exponent n.

Divide the second side of the equation by the known number r; the quotient will be

$$ar^{n-1} + br^{n-2} + cr^{n-3} + \ldots\ldots pr + q = Q,$$

and the remainder s, which thus becomes known. In like manner, dividing the number Q by r, the quotient is

$$ar^{n-2} + br^{n-3} + cr^{n-4} + \ldots\ldots p = Q',$$

and the remainder q. And it is plain, that by thus dividing N, Q, Q′, &c., by r, the successive remainders will be the required coefficients s, q, p, &c.; and the number of divisions, minus 1, will denote the value of n. These coefficients will be no other than the digits or figures which express any number N in the arithmetical scale of notation in which the radix is r; the radix in the common or decimal scale is 10.

For example, suppose it were required to convert the number 17486 from the decimal to the senary scale, that is, to express it in the scale of which the radix is 6, the range of figures in this scale being 0, 1, 2, 3, 4, 5.

Here we have to find the coefficients a, b, c, &c., so as to fulfil the condition

$$a \cdot 6^n + b \cdot 6^{n-1} + c \cdot 6^{n-2} + \ldots\ldots s = 17486.$$

Dividing therefore successively by 6, and noting the several remainders, we have

$$
\begin{array}{rll}
6\underline{)\ 17486} & & \\
6\underline{)\ 2914}, & \text{remainder} = & 2 = s \\
6\underline{)\ 485} & \quad ,, & 4 = q \\
6\underline{)\ 80} & \quad ,, & 5 = p \\
6\underline{)\ 13} & \quad ,, & 2 = c \\
6\underline{)\ 2} & \quad ,, & 1 = b \\
0 & \quad ,, & 2 = a.
\end{array}
$$

Therefore 17486, in the decimal, or denary scale, and which means $1 \,.\, 10^4 + 7 \,.\, 10^3 + 4 \,.\, 10^2 + 8 \,.\, 10 + 6$, when converted into the senary scale is 212542, which means $2 \,.\, 6^5 + 1 \,.\, 6^4 + 2 \,.\, 6^3 + 5 \,.\, 6^2 + 4 \,.\, 6 + 2 = 17486$. In the common or denary scale every digit is increased in a tenfold proportion by being advanced a place to the left; in the senary scale the advance increases the digit in a sixfold proportion; in the quaternary scale in a fourfold proportion; in the ternary, in a threefold; and in the binary, in a twofold.

For ample information on scales of notation the learner may consult Barlow's "Theory of Numbers."

II. To find the value of a vanishing fraction, that is, of a fraction the terms of which vanish for a particular value of some general symbol common to both.

Let $\frac{P}{Q}$ be a fraction such that a certain value a being put for x (a variable quantity supposed here to enter P and Q), it is found that, for such value of x, there results $P = 0$, and $Q = 0$, the fraction then taking the form $\frac{0}{0}$: it is required to find what the true value really is which is concealed under this ambiguous form.

It is plain, that in order that P and Q may simultaneously vanish for $x = a$, the fraction $\frac{P}{Q}$ must be of the form

$\frac{p(x-a)^m}{q(x-a)^n}$, the exponents m, n, one or both, being integral or fractional; P and Q could not vanish together for any value of x, unless for that value there vanished some *factor* common to P and Q. If therefore this common factor be cancelled, the cause of the vague form $\frac{0}{0}$ will be removed, and the true value of the fraction obtained, as in the following examples:—

1. Required the value of $\frac{x^2-a^2}{x-a}$, when $x=a$.

For this value of x, the fraction takes the form $\frac{0}{0}$, a sure sign that the numerator and denominator have a common measure that *also* vanishes for $x=a$. It is easy to see that $x-a$ is this common measure, and that

$$\frac{x^2-a^2}{x-a}=\frac{x+a}{1}=2a, \text{ when } x=a.$$

2. Required the value of $\frac{x-a+\sqrt{(2ax-2a^2)}}{\sqrt{(x^2-a^2)}}$, when $x=a$.

In order to put in evidence the vanishing factor, it will be convenient to write the fraction thus

$$\frac{\sqrt{\{(x-a)(x-a)\}}+\sqrt{\{2a(x-a)\}}}{\sqrt{\{(x+a)(x-a)\}}},$$

in which form it is at once seen that the vanishing factor common to numerator and denominator is $\sqrt{(x-a)}$. Cancelling this common factor, the general fraction becomes $\frac{\sqrt{(x-a)}+\sqrt{2a}}{\sqrt{(x+a)}}$, and the particular case of it for $x=a$ is evidently $\frac{\sqrt{2a}}{\sqrt{2a}}=1$.

3. Required the value of $\frac{1-3x^2+2x^3}{(1-x)^2}$, when $x=1$

Here the vanishing factor, common to numerator and denominator, must evidently be $1 - x$, at least; probably $(1 - x)^2$.

$$1 - x)\ 1 - 3x^2 + 2x^3\ (1 + x - 2x^2 = 1 - x^2 + x - x^2$$
$$\begin{array}{l} 1 - x \\ \hline \quad x - 3x^2 \\ \quad x - x^2 \\ \hline \qquad - 2x^2 + 2x^3 \\ \qquad - 2x^2 + 2x^3 \\ \hline \end{array}$$

The quotient $1 - x^2 + x - x^2$ is clearly divisible also by $1 - x$, the second quotient being $1 + x + x$. Consequently

$$\frac{1 - 3x^2 + 2x^3}{(1 - x)^2} = \frac{1 + 2x}{1} = 3, \text{ when } x = 1$$

4. Required the value of $\dfrac{(a + x)^n - a^n}{x}$, when $x = 0$.

By the Binomial Theorem, the numerator of this fraction is

$$n a^{n-1} x + \frac{n(n-1)}{2} a^{n-2} x^2 + \ldots\ldots \text{ to } n \text{ terms,}$$

therefore dividing by x, the denominator, the general value of the fraction is

$$n a^{n-1} + \frac{n(n-1)}{2} a^{n-2} x + \&c. = n a^{n-1}, \text{ when } x = 0.$$

III. To determine the greatest or least values which algebraical expressions admit of under proposed conditions.

To find what particular value a variable quantity must have in order that the expression into which it enters may be greater or less than it would be for any other value if that variable is a problem belonging to what is called the *maxima* and *minima* of algebraical quantities, and which in its widest extent can be solved only by help of the differential calculus (see the Rudimentary Treatise on that subject). But many questions belonging to maxima and minima can be solved by

aid only of quadratic equations: we shall here present a specimen of such solutions, following the example of Bourdon in his "Algèbre"

1. Divide a given number $2a$ into two parts, such that the product of those parts may be the greatest possible.

Let x be one part, then $2a - x$ is the other; let also y represent the product of these when *greatest:* then we have $x(2a - x) = y$.

$$\therefore 2ax - x^2 = y, \therefore x^2 - 2ax = -y.$$

Solving this quadratic, as if y were known, we have

$$x = a \pm \sqrt{(a^2 - y)} \ldots\ldots (1).$$

Now the greatest value that y can have consistently with a *real* value of x, is evidently $y = a^2$. This, therefore, is the maximum value of the product $2ax - x^2$, that is to say, we must have $2ax - x^2 = a^2$, or (1), $x = a \pm \sqrt{(a^2 - a^2)} = a$. Consequently, the product of the two *equal* parts of a number is always greater than the product of any two *unequal* parts of it.

2. Divide a given number $2a$ into two parts, such that the sum of their squares shall be less than the sum of the squares of any other two parts into which that number may be divided.

Let the parts be x and $2a - x$, and put y for the sum of the squares when this sum is a minimum: we then have

$$x^2 + (2a - x)^2 = y$$

$$\therefore 2x^2 - 4ax + 4a^2 = y$$

$$\therefore x^2 - 2ax = \frac{y}{2} - 2a^2$$

$$\therefore x = a \pm \sqrt{\left(\frac{y}{2} - a^2\right)} \ldots (1).$$

Now the least value that y can have consistently with the reality of x, is evidently such as to render $\frac{y}{2} - a^2 = 0$, because for a less value than this the expression under the radi-

cal sign would be negative, and therefore x would be imaginary, consequently $y = 2a^2$; therefore (1),

$$x = a \pm \sqrt{(a^2 - a^2)} = a,$$

so that, as in the former problem, the number must be divided into *equal* parts.

3. Divide a number $2a$ into two parts, such that the sum of the quotients of each part by the other shall be a minimum.

Putting y for the minimum value, we have

$$\frac{x}{2a - x} + \frac{2a - x}{x} = y, \quad \therefore x^2 + (2a - x)^2 = (2a - x)xy$$

$$2x^2 - 4ax + 4a^2 = (2a - x)xy$$

$$\therefore 2 - \frac{4a^2}{(2a - x)x} = -y$$

$$\therefore x^2 - 2ax = -\frac{4a^2}{2 + y} \quad \therefore x = a \pm \sqrt{\left(a^2 - \frac{4a^2}{2 + y}\right)}.$$

Equating the expression under the radical to zero, we have

$$a^2 - \frac{4a^2}{2 + y} = 0, \quad \therefore 2 + y - 4 = 0$$

$$\therefore y = 2 \quad \therefore x = a \quad \therefore 2a - x = a,$$

so that as before the number must be divided in two equal parts; the sum of the quotients being 2.

4. It is required to determine whether the expression $\dfrac{4x^2 + 4x - 3}{6(2x + 1)}$ admits of a maximum or a minimum.

As before, put $\dfrac{4x^2 + 4x - 3}{6(2x + 1)} = y$

$$\text{then } 4x^2 - 4(3y - 1)x = 6y + 3$$

$$\therefore x = \frac{3y - 1}{2} \pm \frac{1}{2}\sqrt{(9y^2 + 4)}.$$

Now whatever real value be given to y, the quantity under the radical sign will always be positive; so that x will continue real, however small or however great y be taken; therefore there is no limitation as to the smallness or largeness of y; that is, there is no maximum or minimum value of the proposed expression.

IV. To find the number of shot in a pile, whether square, triangular, or rectangular.

In arsenals and fortresses, cannon balls and shells are arranged in piles with either square, triangular, or rectangular bases.

A square pile is formed by a succession of horizontal courses one above another, each course forming a square, the pile being crowned by a single shot, the sides of the successive courses decreasing also by a single shot in regular order from bottom to top.

A triangular pile is made up of horizontal triangular courses, the sides of the successive courses decreasing by a single shot, and the pile terminating, as in the square pile, by one shot at top.

In both these piles the sides or faces are equilateral triangles, the shot in each face forming an arithmetical progression, the first of which is 1, and the last the number of shot in the bottom row.

If to one of the sloping faces of the square pile a new and equal face of shot be added, the whole pile will of course terminate in *two* shot; if another face be added it will terminate in *three* shot, and so on; and in this way the rectangular pile may be regarded as formed; the number of triangular faces of shot added to the square pile being thus always one less than the number of shot in the top row or edge; or, which is the same thing, this number is equal to the difference between the number of shot in the longer side and the number in the shorter side of the rectangular base.

1. How many shot are there in a square pile which has 8 shot in the bottom row?

Commencing at the top shot, and adding together the several courses, we have

$$1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 = 204,$$

or by the formula at page 145 *

$$1 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

which for $n = 8$, gives $\frac{8 \times 9 \times 17}{6} = 4 \times 3 \times 17 = 204.$

2. How many shot are there in an oblong or rectangular pile, the number of shot in one side of the base being 16, and the number in the other side 7?

By the above formula the number of shot in the square pile, the side of whose base has 7 shot, is

$$\frac{7 \times 8 \times 15}{6} = 7 \times 4 \times 5 = 140.$$

The number of shot in a face of this pile is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$; and as 9 such faces have been added to the square pile, $28 \times 9 = 252$ shot have been added to that pile to form the rectangular pile.

$\therefore\ 140 + 252 = 392$, the number of shot in the pile.

It is plain that, as in this example, the number of shot in an oblong pile will always be obtained by adding to the square pile, the side of whose base is n—the shortest side of the oblong base—the arithmetical progression $1 + 2 + 3 \ldots + n$ multiplied by $m - n$, m being the longest side of the oblong base.

The general formula for the shot in an oblong or rectangular pile is therefore

$$\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)(m-n)}{2} =$$

$$\frac{n(n+1)\{2n + 1 + 3m - 3n\}}{6} =$$

$$\frac{n(n+1)(3m - n + 1)}{6}.$$

* The truth of this general expression for the sum of the squares of the natural numbers, is established at Article V. of this Appendix.

3. How many shot are there in an oblong pile, the shorter side of the base of which contains 15, and the longer side 35?

Here $n = 15$, and $m = 35$.

$\therefore \dfrac{15 \times 16 \times 91}{6} = 5 \times 8 \times 91 = 3640$, the number of shot.

4. How many shot are there in a triangular pile, of which each side of the base contains 8 shot?

It is obvious that, commencing at the bottom course, the shot in the several courses are given by the following arithmetical progressions:—

1st course	$1 + 2 + 3 + \ldots + 8$	$= 9 \times 4$	$= 36$
2nd „	$1 + 2 + 3 + \ldots + 7$	$= 8 \times 3\frac{1}{2}$	$= 28$
3rd „	$1 + 2 + 3 + \ldots + 6$	$= 7 \times 3$	$= 21$
4th „	$1 + 2 + 3 + \ldots + 5$	$= 6 \times 2\frac{1}{2}$	$= 15$
5th „	$1 + 2 + 3 + 4$	$= 5 \times 2$	$= 10$
6th „	$1 + 2 + 3$	$= 4 \times 1\frac{1}{2}$	$= 6$
7th „	$1 + 2$		$= 3$
8th „	... ...	... ...	1
			120

the number of shot.

It is plain that the n layers or courses in a triangular pile is always the sum of n terms of the series.

$$1 + 3 + 6 + 10 + 15 + 21 + 28 + \&c.;$$

or, taking the terms in pairs, if there be an *even* number of them, the sum will be $\frac{n}{2}$ terms of the series $4 + 16 + 36 +$ &c.; or $4\,(1 + 2^2 + 3^2 + \&c.$, to $\frac{n}{2}$ terms).*

* That the sum of any two consecutive terms in the series of *triangular numbers*—as these numbers are, for obvious reasons, called—is always a square number, may be proved in general terms, as follows:—

The nth triangular number being the sum of the arithmetical series

$$1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots + n$$

is $\dfrac{n\,(n+1)}{2}$, consequently the preceding, or $n - 1$th term, is $\dfrac{(n-1)\,n}{2}$;

The expression for this sum by the formula above is

$$4 \times \frac{1}{6} \cdot \frac{n}{2}\left(\frac{n}{2} + 1\right)(n + 1) = \frac{n(n + 1)(n + 2)}{6}.$$

But if n be an odd number, then taking the first $n - 1$ terms of the series, as above, the sum of *them*, by the formula just deduced, putting $n - 1$ for n, is $\frac{(n - 1)n(n + 1)}{6}$; to which the nth term, or

$$1 + 2 + 3 + 4 + \ldots\ldots + n = \frac{n}{2}(n + 1)$$

must be added, to get the sum of the entire n terms.

$$\text{Now } \frac{(n - 1)n(n + 1)}{6} + \frac{n(n + 1)}{2} =$$

$$\frac{n(n - 1)(n + 1) + 3n(n + 1)}{6} =$$

$$\frac{\{n(n - 1) + 3n\}(n + 1)}{6} = \frac{n(n + 1)(n + 2)}{6}.$$

Hence the expression for the sum of the n courses in a triangular pile of shot is the same whether n be even or odd.

Applying this formula to the foregoing example, where $n = 8$, we have $\frac{8 \times 9 \times 10}{6} = 4 \times 3 \times 10 = 120$, the number of shot in the pile.

V. To find the sum of the series

$$1 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots\ldots + n^2,$$

by the method of indeterminate coefficients.

Put $1 + 2^2 + 3^2 + \ldots. + n^2 = A + Bn + Cn^2 + Dn^3 +$ &c.: the object is to determine the numerical values of A, B, C, &c.

---

and $\frac{(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{2n^2}{2} = n^2.$

Hence the nth term, added to the preceding term in the series of triangular numbers, is n^2.

If n be changed into $n+1$, the preceding identical equation becomes

$$1 + 2^2 + 3^2 + \ldots\ldots + (n+1)^2 =$$
$$A + B(n+1) + C(n+1)^2 + D(n+1)^3 +, \&c.$$

Subtracting the former from this there remains

$$n^2 + 2n + 1 = B + 2Cn + C + 3Dn^2 + 3Dn + D,$$

the subsequent terms, involving the coefficients E, F, &c., being omitted; because, since these coefficients would be connected with n^3, n^4, &c., they would all be zero, seeing that these powers of n are absent from the first member of the identical equation. Hence, equating the coefficients of the like powers of n, we have the conditions

$$3D = 1 \quad \therefore \quad D = \frac{1}{3}$$

$$2C + 3D = 2 \quad \therefore \quad C = \frac{1}{2}$$

$$B + C + D = 1 \quad \therefore \quad B = \frac{1}{6};$$

A is evidently 0, and might have been suppressed at the outset, because when $n = 0$ in the first identity, that identity becomes $0 = A$.

$$\therefore 1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$$
$$= \frac{n + 3n^2 + 2n^3}{6}$$
$$= \frac{n(1 + 3n + 2n^2)}{6}$$
$$= \frac{n(n+1)(2n+1)}{6}$$

VI. Binomial Equations—Reciprocal Equations.

A binomial equation is an equation consisting of but two terms, and belonging to the general form $x^n + a = 0$, where a may be either positive or negative. This form may be reduced to the more simple one $x^n \pm 1 = 0$ by dividing it by a,

or $a^{\frac{n}{n}}$; for we then have $\left(\frac{z}{\sqrt[n]{a}}\right)^n \pm 1 = 0$, which, by putting x for $\frac{z}{\sqrt[n]{a}}$ becomes $x^n \pm 1 = 0$.

Let $n = 2$: then the solutions of the equations $x^2 - 1 = 0$, and $x^2 + 1 = 0$, are $x = \pm 1$, and $x = \pm\sqrt{-1}$.

Let $n = 3$: then since $x^3 - 1 = (x-1)(x^2 + x + 1)$, the solution of the equation $x^3 - 1 = 0$ becomes the solution of the two equations

$$x - 1 = 0, \text{ and } x^2 + x + 1 = 0;$$

that is, the values of x are

$$x = 1,\ x = \frac{-1 + \sqrt{-3}}{2},\ x = \frac{-1 - \sqrt{-3}}{2}.$$

These, therefore, are the *three* values of the cube root of 1. If instead of 1 the second term of the binomial were any other number a, then we should have to multiply each of these cube roots of unity by the arithmetical cube root of a. For instance, suppose a were 8, then we should have

$$\sqrt[3]{8} = 2, \text{ or } -1 + \sqrt{-3}, \text{ or } -1 - \sqrt{-3}.$$

In like manner $\sqrt[3]{27} = 3$, or $\frac{3}{2}(-1 + \sqrt{-3})$,

or $\frac{3}{2}(-1 - \sqrt{-3})$.

If either of these roots be cubed, the result will of course be the power under the radical sign on the left.

If the equation had been $x^3 + 1 = 0$, then since one root is evidently $x = -1$, $x + 1$ is a factor of $x^3 + 1$, so that $x^3 + 1 = (x+1)(x^2 - x + 1)$. Consequently the three roots of the proposed binomial equation are given by the roots of

$$x + 1 = 0, \text{ and } x^2 - x + 1 = 0,$$

which are $x = -1$, $x = \frac{1 + \sqrt{-3}}{2}$, $x = \frac{1 - \sqrt{-3}}{2}$.

Let $n = 4$: then since $x^4 - 1 = (x^2 - 1)(x^2 + 1)$, the solution of the equation $x^4 - 1 = 0$ becomes the solution of the two equations

$$x^2 - 1 = 0, \text{ and } x^2 + 1 = 0;$$

so that the values of x are

$$x = 1,\ x = -1,\ x = \sqrt{-1},\ x = -\sqrt{-1}.$$

These, therefore, are the *four* values of the fourth root of 1. If the fourth root of 16 be required, then, omitting no value that has claim to be called a fourth root, we should have

$$\sqrt[4]{16} = 2, \text{ or } -2, \text{ or } 2\sqrt{-1}, \text{ or } -2\sqrt{-1}.$$

If the equation had been $x^4 + 1 = 0$, the roots would have been those of the equations

$$x^2 + \sqrt{-1} = 0, \text{ and } x^2 - \sqrt{-1} = 0,$$

$$\text{because } (x^2 + \sqrt{-1})(x^2 - \sqrt{-1}) = x^4 + 1,$$

$$\therefore x = \pm\sqrt{(-\sqrt{-1})}, \text{ and } \pm\sqrt{(\sqrt{-1})}.$$

The principle here illustrated is perfectly general: the nth root of a number always has n different values; but to determine the expressions for these roots when n is a high number, it is desirable to avail ourselves of the aid of Trigonometry. (See the Rudimentary Treatise on that subject.)

Sometimes it happens that the coefficients of an equation are such that the equation would remain substantially unaltered, though the unknown quantity in it were replaced by its reciprocal. In fact, the binomial equation $x^n + 1 = 0$ is of such a form; because though $\frac{1}{x}$ be put for x, the equation preserves the same meaning, for $\frac{1}{x^n} + 1 = 0$, reduces to $1 + x^n = 0$; which is the same as the original. In like manner $x^n - 1 = 0$, $\frac{1}{x^n} - 1 = 0$, $1 - x^n = 0$, $-1 + x^n - 0$, are all virtually the same.

Whenever the coefficients of an equation are such, that whether taken in order from left to right, or from right to left, they give *the same* series of numbers, the equation will remain substantially unaltered, though $\frac{1}{x}$ be put for x; thus in the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0,$$

the coefficients in which furnish the same series 1, a, b, a, 1, whether taken from left to right, or from right to left, if $\frac{1}{x}$ be put for x, we shall have

$$\frac{1}{x^4} + \frac{a}{x^3} + \frac{b}{x^2} + \frac{a}{x} + 1 = 0;$$

that is $$1 + ax + bx^2 + ax^3 + x^4 = 0,$$

the same as the original.

Such equations are called *Reciprocal Equations*, and sometimes *Recurring Equations*.

If the signs of the equal coefficients were unlike instead of like, and the equation of an *even* degree—like that above—then the middle term (bx^2) must be absent for the equation to be a reciprocal equation: for the coefficients in $x^4 + ax^3 + bx - ax - 1 = 0$, when taken in direct and reverse order are

$$1, +a, +b, -a, -1 \text{ ...... (1)}$$

$$\text{and } -1, -a, +b, +a, +1;$$

or, multiplying each of the latter by -1, they are

$$1, +a, -b, -a, -1 \text{ ...... (2)};$$

but the series (1), (2) are not the same unless $b = 0$. And such must always be the case when the equation is of an even degree, because every such equation, in its complete state, consists of an odd number of terms, so that the middle coefficient will appear in the two series of coefficients with contrary signs, after the signs of the others have been made like.

But in an equation of an odd degree, where the number of terms is always even when none are omitted, it matters not whether the equal coefficients have like or unlike signs.

The following are examples of the solution of reciprocal equations.

1. Let $x^4 - 3x^3 - 2x^2 - 3x + 1 = 0$,

dividing by x^2, $x^2 - 3x - 2 - \frac{3}{x} + \frac{1}{x^2} = 0$;

that is, $\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 2 = 0 \ldots\ldots (1)$.

Now, $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$; $\therefore x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$;

$$\therefore (1), \left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) = 4,$$

$$\text{or, } y^2 - 3y = 4; \;\therefore\; y = 4, \text{ or } -1;$$

$$\therefore\; x + \frac{1}{x} = 4, \text{ or } x + \frac{1}{x} = -1;$$

and from these equations we get for x the four values,

$$2 + \sqrt{3},\ 2 - \sqrt{3},\ -\frac{1 + \sqrt{3}}{2},\ -\frac{1 - \sqrt{3}}{2}.$$

2. Let $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$.

It is plain that $x = 1$ satisfies this equation, so that dividing by $x - 1$, we have

$$x^4 + x^3 - 4x^2 + x + 1 = 0;$$

$$\therefore\; \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 0;$$

$$\therefore\; y^2 + y = 6; \;\therefore\; y = \frac{-1 \pm 5}{2} = 2, \text{ or } -3;$$

$$\therefore\; x + \frac{1}{x} = 2, \text{ or } x + \frac{1}{x} = -3$$

From the first of these $x^2 - 2x + 1 = 0$, that is $(x - 1)^2 = 0$; $\therefore x = 1, x = 1$.

From the second, $x = \frac{-3 \pm \sqrt{5}}{2}$: hence the five roots are

$$1, 1, 1, \frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}.$$

As, in these examples, every reciprocal equation of an *even* degree $(2n)$ may be solved by equations of the degree n; and every reciprocal equation of an *odd* degree $(2n + 1)$ may also be reduced to others of the degree n, for when the degree is odd, one root of the equation is always $x = 1$, or $x = -1$, according as its final term, is $+ 1$, or $- 1$.

Since not only $x = r$ (r being put for any root) solves the reciprocal equation, but also $x = \frac{1}{r}$, as already seen, it follows that the only differing roots enter in *pairs*, each pair consisting of two roots that are the reciprocals of each other. Thus in Example 1, above,

$$2 - \sqrt{3} = \frac{1}{2 + \sqrt{3}}; \text{ and } \frac{1 - \sqrt{3}}{2} = \frac{2}{1 + \sqrt{3}},$$

so that $2 + \sqrt{3}$, $2 - \sqrt{3}$ are reciprocals, as also $-\frac{1 + \sqrt{3}}{2}$, $-\frac{1 - \sqrt{3}}{2}$.

In like manner in Example 2, $\frac{-3 - \sqrt{5}}{2}$ is the reciprocal of $\frac{-3 + \sqrt{5}}{2}$. And, omiting the root $+ 1$, or $- 1$, in an equation of an odd degree, we may infer generally that for each root there is always another which is the reciprocal of it, and which may be paired with it, as above. Hence the propriety of calling these equations *reciprocal equations*.

MISCELLANEOUS EXAMPLES.

PARTLY FROM THE CAMBRIDGE EXAMINATION PAPERS, 1853–54.

1. Find two numbers in the ratio of 8 : 5, such that their product may be 360. Ans. 24 and 15.

2. Two persons, A and B, can together perform a piece of work in 16 days. They work together for 4 days, when A is called off, and B left to finish it, which he does in 36 days more. In what time could each separately do the whole?
Ans. A in 24 days, B in 48 days

3. What quadratic equation is that of which the roots are 3 and 5? Ans. $x^2 - 8x + 15 = 0$

4. What value must c have so that the roots of the equation $2x^2 + 8x + c = 0$ may be equal?
Ans. $c = 8$; the roots then being each $= -2$

5. If x be real, prove that $x^2 - 8x + 22$ can never be less than 6.

6. Prove that $(a + b)(b + c)(a + c)$ must be greater than $8\,abc$, unless a, b, c, are all equal.

7. How many changes can be rung upon 12 bells?
Ans. 479,001,600.

8. How many permutations can be formed out of the letters in the word Mississippi. Ans. 34,650

9. Find the roots of the equation $x^4 + 1 = 0$.
Ans. $\dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}$ and $\dfrac{-1 \pm \sqrt{-1}}{\sqrt{2}}$.

10. A and B engaged to reap equal quantities of wheat. A began half-an-hour before B, but they both rested from 12 to 1 o'clock, having finished just half the work. B having completed his part of the work, left off at 7 o'clock, but A was obliged to remain till ¼ to 10 to complete his part. At what hour did A begin? Ans. ½ past 4 in the morning

11. A ship, with a crew of 175 men, set sail with water enough for the voyage; but at the end of 30 days the scurvy began to carry off 3 men daily, and a storm protracted the voyage 3 weeks. But the water still just lasted the voyage. Required the time of passage. Ans. 79 days.

12. Prove that $(x + y)^7 - (x^7 + y^7)$ is exactly divisible by $(x^2 + xy + y^2)^2$.

13. How many combinations may be made of 96 things out of 100? Ans. 3,921,225.

14. Prove that in a geometrical progression the sum of any *odd* number of terms will always exactly divide the sum of their squares.

15. If the second term of an arithmetical progression be a mean proportional between the first and fourth, prove that the sixth will be a mean proportional between the fourth and ninth.

16. The fourth term of a geometrical progression is 9, and the seventh term 15, find the series.

$$\text{Ans. } \frac{27}{5} + \frac{27}{5}\sqrt[3]{\frac{5}{3}} + \frac{27}{5}\sqrt[3]{\frac{25}{9}} + 9 + 9\sqrt[3]{\frac{5}{3}}$$
$$+ 9\sqrt[3]{\frac{25}{9}} + 15 + \&c$$

17. Insert five geometrical means between 2 and $\frac{1}{32}$.

$$\text{Ans. } \pm 1, \frac{1}{2}, \pm \frac{1}{4}, \frac{1}{8}, \pm \frac{1}{16}.$$

18. How many terms of the series $9 + 7 + 5 +$ &c. will amount to 24? Ans. Either *four* terms, or *six* terms.

19. How many changes may be rung by 5 bells out of 6? Ans. 720.

20. A person engages to entertain 10 friends in parties of 6 at a time, as many days as it is possible to form different combinations of 6. Required the number of days. Ans. 210

21. Given the equations

$$\left.\begin{array}{r} x^2 + y^2 = r^2 \\ ax + by = r^2 \end{array}\right\} \text{ to find expressions for } x \text{ and } y.$$

$$\text{Ans. } x = \frac{ar^2}{a^2 + b^2} \mp \frac{br}{a^2 + b^2} \sqrt{(a^2 + b^2 - r^2)}.$$

$$y = \frac{ar^2}{a^2 + b^2} \pm \frac{br}{a^2 + b^2} \sqrt{(a^2 + b^2 - r^2)}.$$

22. Show how

$$\frac{x}{6} + \frac{x^2}{2} + \frac{x^3}{3} \text{ is to be converted into } \frac{x(x+1)(2x+1)}{6}$$

23. A person has spirits at a shillings a gallon, and at b shillings a gallon, and he wishes to make a mixture of d gallons that shall be worth c shillings a gallon: how much of each must he take?

Ans. $\frac{c - b}{a - b} d$ gallons of the first,

$\frac{a - c}{a - b} d$ „ „ second

24. If from a vessel of wine containing a gallons, b gallons be drawn off, and the vessel then filled up with water, and if this operation be continued, b gallons being drawn off and the quantity replaced by water n times successively, what quantity of wine will remain in the vessel after the nth operation?

Ans. $\frac{(a - b)^n}{a^{n-1}}$ gallons.

25. For what value of x does the expression $\frac{(a + x)(b + x)}{x}$ become the smallest possible? Ans. $x = \sqrt{ab}$.

26. How many shot are there in a square pile, the number in the bottom row being 30? Ans. 9455.

27. How many shot are there in a triangular pile, the number in the bottom row being 30? Ans. 4960.

28. Determine the value of $\frac{2x^3 - 5x^2 - 4x + 12}{x^3 - 12x + 16}$, when $x = 2$.

Ans. $\frac{7}{6}$.

29. Determine the value of $\frac{x^n}{a - \sqrt[n]{(a^n - x^n)}}$, when $x = 0$.

Ans. na^{n-1}.

30. Convert the number 1810 into its equivalent in the ternary scale of notation. Ans. 2111001.

31. Into how many different triangles may a polygon of n sides be divided by means of diagonals, one side at least of the polygon always being a side of a triangle?

Ans. $\frac{n(n-1)(n-2)}{6}$.

32. How many diagonals can be drawn in a polygon of n sides?

Ans. $\frac{n(n-3)}{2}$.

33. A telegraph has 4 arms, and each arm is capable of 5 distinct positions; find the total number of signals which can be made with the telegraph. Ans. 1295.

34. A merchant bought a cask of spirits for 48*l.*, and sold a quantity exceeding three-fourths of the whole by two gallons at a profit of 25*l.* per cent. He afterwards sold the remainder at such a price as to clear 60*l.* per cent. by the whole transaction; but if he had sold the whole quantity at the latter price, he would have gained 175*l.* per cent. How many gallons were contained in the cask?* Ans. 120 gallons.

* This question may be solved by the principles of common arithmetic. It was proposed a few years ago to Henri Mondeux, "the Shepherd of Touraine," a youth of most remarkable powers of calculation without the aid of pen or pencil. The following notice is extracted from "The Mathematician," vol. ii. p. 218:—

"This wonderful youth having lately visited Jersey to exhibit his extraordinary powers of mental calculation, the above exercise was proposed to him, and a nearly instantaneous answer was given. At the request of Mr. Godfray he wrote out his solution, of which the following is the translation:—

"He sells the first portion at a profit of 25*l.* per cent., and the last at

35. It is required to determine A and B so that the equation

$$\frac{cx + d}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

may always be true, whatever be the value of x.

$$\text{Ans.} \begin{cases} A = \dfrac{ac-d}{a-b}. \\ B = \dfrac{d-bc}{a-b}. \end{cases}$$

36. Given the equation

$$4x^4 - 12x^3 - 35x^2 + 66x - 8303 = 0$$

to find x by quadratics.

$$\text{Ans. } x = \frac{3 \pm \sqrt{(97 \pm 144\sqrt{26})}}{4}.$$

37. Prove that the two series following are identical, namely,

$$a^n \left\{1 + n\frac{b}{a+b} + \frac{n(n+1)}{1 \cdot 2}\frac{b^2}{(a+b)^2} + \&c.\right\} =$$

$$b^n \left\{1 + n\frac{a}{a+b} + \frac{n(n+1)}{1 \cdot 2}\frac{a^2}{(a+b)^2} + \&c.\right\}$$

where n is any number whatever.

38. Prove that the following expressions are identical, namely,

$$(n+1)(n+2) \ldots\ldots (n+n) = 2^n \times 1 \cdot 3 \cdot 5 \ldots\ldots (2n-1);$$

and thence that the sum of the numerical coefficients in the

175*l.* per cent., and gains 60*l.* per cent. on the whole. The first profit is less than the mean profit by 35*l.* per cent., and the second is greater by 115*l.* per cent.; he has therefore sold 115 parts of the first against 35 of the second; that is, the first portion sold was the $\frac{115}{150}$ of the whole cask, and the last the $\frac{35}{150}$; but the first portion was $\frac{3}{4}$ of the cask and 2 gallons more; and the difference between $\frac{115}{150}$ or $\frac{23}{30}$ and $\frac{3}{4}$ is $\frac{1}{60}$; therefore 2 gallons is the $\frac{1}{60}$ of the cask; that is, the cask contained 120 gallons."

development of any positive integral power of a binomial, that is,

$$1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{2 . 3} + \&c.$$

is equal to

$$\frac{(n+1)(n+2) \ldots\ldots (n+n)}{1 . 3 . 5 \ldots\ldots (2n-1)}$$

$$= \frac{(n+1)(n+3) \ldots\ldots (n+n)}{1 . 3 . 5 \ldots\ldots n} \text{ when } n \text{ is odd}$$

$$= \frac{(n+2)(n+4) \ldots\ldots (n+n)}{1 . 3 \ 5 \ldots\ldots (n-1)} \text{ when } n \text{ is even.}$$

39. It is required to prove by algebra the principle known in arithmetic as "the method of casting out the nines," and which is employed to test the accuracy of a common multiplication process.

40. It is required to prove the following, which has been called the *Reciprocal Theorem*:—

$$x^n + \frac{1}{x^n} = \left(x + \frac{1}{x}\right)^n - n\left(x + \frac{1}{x}\right)^{n-2} + \frac{n(n-3)}{1 . 2}$$
$$\left(x + \frac{1}{x}\right)^{n-4} - \frac{n(n-4)(n-5)}{1 . 2 . 3}\left(x + \frac{1}{x}\right)^{n-6} + \&c.$$

where n is any positive whole number whatever.

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